

# Electromagnetic pulses: propagation & properties<sup>1</sup>

## Propagation equation, group velocity, group velocity dispersion

An electric field wave packet propagating in a laser beam along the  $z$  axis can be described as

$$E(\mathbf{r}, t) = \frac{1}{2} a(z, t) F(x, y, z) e^{i(k_0 z - \omega_0 t)} + c.c. \quad (\text{VIII-18})$$

where  $a(z, t)$  is the slowly-varying temporal envelope,  $F(x, y, z)$  is the transverse amplitude distribution of the light wavepacket (henceforth briefly: pulse),  $\nu_0 = \omega_0/2\pi$  is the *carrier frequency* and  $k_0 = \omega_0 n(\omega_0)/c$  determines the *carrier wavelength* (as  $\lambda_0 = 2\pi/k_0$ ) of the wavepacket. In the absence of an excessively broad bandwidth and strong nonlinearities,  $F(x, y, z)$  can be well approximated with the distribution valid for a monochromatic laser beam at  $\omega_0$  and can be separated out from the equation for  $a(z, t)$ . Under these circumstances,  $a(z, t)$  obeys the following *pulse propagation equation* (Eq. IV-127 in Chapter IV-4):

$$\left[ \frac{\partial^2}{\partial z^2} + 2ik_0 \left( \frac{\partial}{\partial z} + k_1 \frac{\partial}{\partial t} \right) - (k_1^2 + k_0 k_2) \frac{\partial^2}{\partial t^2} \right] a(z, t) = \frac{2}{\epsilon_0 c^2} \frac{\partial^2 P^{NL}(z, t)}{\partial t^2} e^{-i(k_0 z - \omega_0 t)} \quad (\text{VIII-19})$$

where the parameters  $k_0, k_1, k_2$  are defined by the Taylor expansion of the wave vector

$$k(\omega) = \frac{\omega n(\omega)}{c} = k_0 + k_1(\omega - \omega_0) + \frac{1}{2} k_2(\omega - \omega_0)^2 + \frac{1}{6} k_3(\omega - \omega_0)^3 + \frac{1}{24} k_4(\omega - \omega_0)^4 + \dots \quad (\text{VIII-20})$$

and the effects of the higher-order terms proportional to  $k_3, k_4, \dots$  was neglected in (VIII-19)

## Slowly-varying amplitude approximation

With the *slowly-varying amplitude approximation in space & in time*, respectively<sup>2</sup>

$$\left| \frac{1}{k_0} \frac{\partial}{\partial z} a \right| \ll |a| \quad ; \quad \left| \frac{1}{\omega_0} \frac{\partial}{\partial \tau} a \right| \ll |a| \quad (\text{VIII-21a,b})$$

<sup>1</sup> The majority of figures and several derivations have been adapted from the *Ultrafast Optics* lecture course of Prof. Rick Trebino at the Georgia Institute of Technology, Atlanta.

<sup>2</sup> Note that the nature of these approximations is very different: whilst Eq. (VIII-21a) holds if the variation of the electric field amplitude and hence the pulse shape is small upon travelling a distance equal to the carrier wavelength, Eq. (VIII-21b) expresses that the pulse is long enough so that its electric field amplitude varies – at any position – within the carrier oscillation period. Whereas the first approximation is to be fulfilled by the propagation medium, the second one is to be met by the pulse itself.

and transformation to the “retarded” frame of reference

$$\tau = t - k_1 z = t - \frac{z}{v_g} \quad (\text{VIII-22})$$

moving with a speed  $v_g = 1/k_1$  we obtain<sup>3</sup>

$$\frac{\partial}{\partial z} a(\tau, z) = -\frac{ik_2}{2} \frac{\partial^2}{\partial \tau^2} a + \frac{i\omega_0 \epsilon_0 n n_2}{2} |a|^2 a \quad (\text{VIII-23})$$

where we neglected terms of third order and higher in (VIII-20) and assumed that the dominant nonlinearity is the optical Kerr effect, i.e. and intensity dependent refractive index

$$\Delta n(t) = n_2 I(t) \quad (\text{VIII-24})$$

Equation VIII-23 describes optical pulse propagation in a dispersive & nonlinear medium, and due to the formal analogy, has been dubbed the *nonlinear Schrödinger equation*. It is of key importance for optical fibre communication and, as we shall see, for describing femtosecond pulse formation in mode-locked solid state lasers.

In (VIII-23)  $k_2 = 0$  and  $n_2 = 0$  implies  $\partial a(\tau, z) / \partial z = 0$ , that is the pulse propagates undistorted with its peak “locked” to  $\tau = 0$  all the time. That is, the pulse propagates at a speed of  $v_g$

$$k_1 = \left( \frac{dk(\omega)}{d\omega} \right)_{\omega_0} = \frac{1}{v_g} \quad (\text{VIII-25})$$

which is therefore referred to as *group velocity* and consequently

$$k_2 = \left( \frac{d^2 k(\omega)}{d\omega^2} \right)_{\omega_0} = \frac{d}{d\omega} \left( \frac{1}{v_g} \right)_{\omega_0} \quad (\text{VIII-26})$$

is the variation of this quantity with frequency, the *group velocity dispersion*. Currently neglected, higher-order terms in the expansion of the wave vector introduce higher-order derivatives with respect to  $\tau$  in the pulse propagation equation and must not be ignored if either  $k_2$  is small or the spectral width of the wavepacket is large.

<sup>3</sup> For derivation see Chapter IV-4.

Note that the group velocity

$$\frac{1}{v_g} = \left. \frac{dk}{d\omega} \right|_{\omega_0} = \frac{n(\omega_0)}{c} + \frac{\omega_0}{c} \left. \frac{dn}{d\omega} \right|_{\omega_0} \quad (\text{VIII-27})$$

is equal to the *phase velocity*

$$\frac{1}{v_{ph}} = \left. \frac{k}{\omega} \right|_{\omega_0} = \frac{k_0}{\omega_0} = \frac{n(\omega_0)}{c} \quad (\text{VIII-28})$$

if and only if  $\left. \frac{dn}{d\omega} \right|_{\omega_0} = 0$ .

### Frequency-domain description of electromagnetic pulses

The electric field of an electromagnetic wavepacket can be expressed in terms of its Fourier components as

$$E(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(z, \omega) e^{-i\omega t} d\omega, \quad \text{where} \quad E(z, \omega) = \int_{-\infty}^{\infty} E(z, t) e^{i\omega t} dt$$

$E(z, \omega)$  describes the wavepacket as uniquely as  $E(z, t)$  does.  $E(z, \omega)$  is related to the Fourier-transform of the complex field amplitude

$$a(z, \omega) = \int_{-\infty}^{\infty} a(z, t) e^{i\omega t} dt \quad (\text{VIII-29})$$

as

$$E(z, \omega) = E_+(z, \omega) + E_-(z, \omega) = \frac{1}{2} a(z, \omega - \omega_0) e^{ik_0 z} + \frac{1}{2} a^*(z, \omega + \omega_0) e^{-ik_0 z} \quad (\text{VIII-30})$$

This equation reveals that the negative and positive-frequency portion of  $E(z, \omega)$  contains redundant information. Therefore for reconstruction of the pulse,  $E(z, t)$ , the positive-frequency portion of  $E(z, \omega)$ ,  $E_+(z, \omega)$ , centred at  $\omega_0$  is sufficient just as the Fourier transform of the complex field amplitude  $a(z, \omega)$ , which centres at  $\omega = 0$  as shown in Fig. VIII-12 for  $z = 0$ .

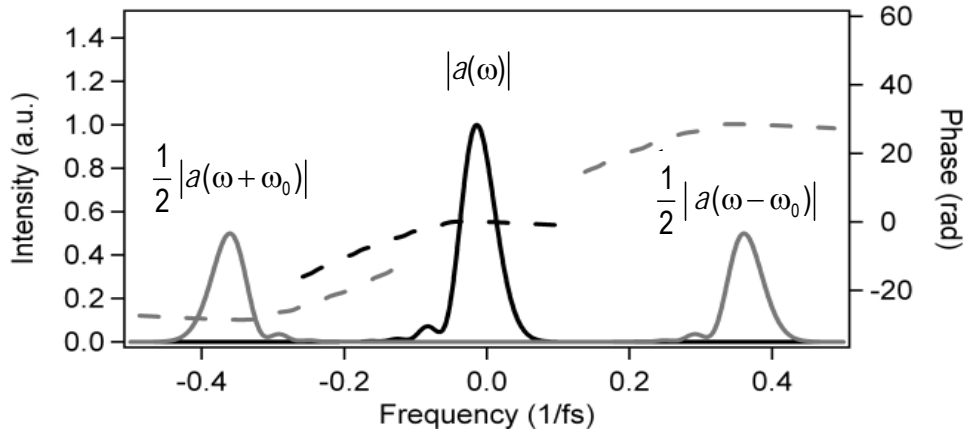


Fig. VIII-12 The full and dashed black lines show  $|a(\omega)|$  and its phase, respectively, whereas the grey lines depict the modulus and phase of  $E(\omega)$ .

The frequency-domain description is particularly helpful if the wavepacket propagates through a linear medium ( $P_{NL} = 0$  in Eq. VIII-19). Under these circumstances, each frequency component  $E(z, \omega)$  of the wavepacket  $E(z, t)$  evolves during propagation just as the amplitude of a plane wave of the same frequency:  $E(z, \omega) = E(0, \omega) e^{ik(\omega)z}$  so that the complex spectral amplitudes at  $z = 0$  and  $z = L$  are connected as

$$E(L, \omega) = E(0, \omega) e^{i\varphi(\omega)} \quad (\text{VIII-32})$$

where the phase shift  $\varphi(\omega) = k(\omega)L$  accumulating over a distance  $L$  can be expanded just as  $k(\omega)$  :

$$\begin{aligned} \varphi(\omega) = k(\omega)L = & \varphi_0 + \varphi_1(\omega - \omega_0) + \frac{1}{2}\varphi_2(\omega - \omega_0)^2 + \\ & + \frac{1}{6}\varphi_3(\omega - \omega_0)^3 + \frac{1}{24}\varphi_4(\omega - \omega_0)^4 + \dots \end{aligned} \quad (\text{VIII-33})$$

Clearly, the expansion coefficients  $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \dots$  are related to those in (VIII-20):  $k_0, k_1, k_2, k_3, \dots$

$$\varphi_0 = k(\omega_0)L = k_0L \quad (\text{VIII-34})$$

is the “absolute” phase shift, which is added to all frequency components of the wavepacket. It is equal to the shift suffered by a monochromatic wave at the carrier frequency  $\omega_0$ .

Similarly

$$\varphi_1 = \left. \frac{d\varphi}{d\omega} \right|_{\omega_0} = k_1 L = \frac{L}{v_g} = T_g, \quad (\text{VIII-35})$$

which is the time it takes the pulse to travel a distance  $L$ , termed the *group delay* and

$$\varphi_2 = \left. \frac{d^2\varphi}{d\omega^2} \right|_{\omega_0} = \left. \frac{dT_g}{d\omega} \right|_{\omega_0} \quad (\text{VIII-36})$$

measures the variation of this delay with  $\omega$  and is called the *group-delay dispersion*.

The frequency-domain description of light pulse propagation through linear media is now as simple as follows: the frequency spectrum of the output pulse is obtained from that of the input pulse by (VIII-32) and inverse Fourier transform yields the output waveform (Fig. VIII-13). The procedure is equivalent to solving the differential equation (VIII-19) for  $R_{NL} = 0$ , but is much more straightforward.

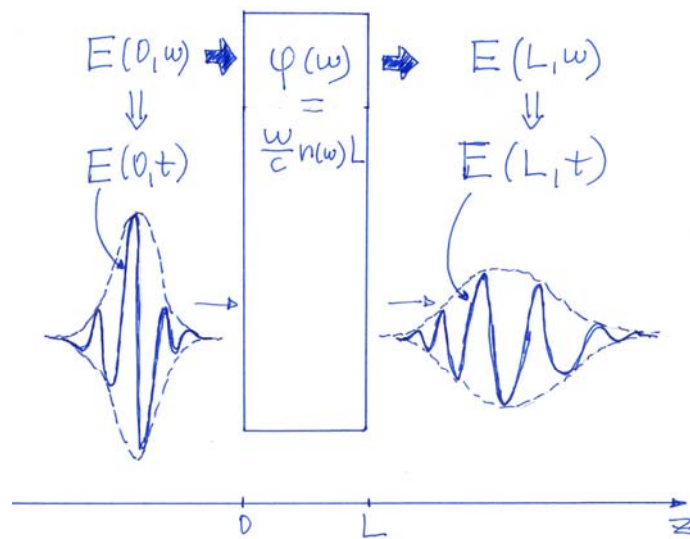


Fig. VIII-13

### Physical implication of the spectral phase shift $\varphi(\omega)$

It is the phase shift, the individual spectral components of a wavepacket suffer upon propagation. Let us see how the linear term in the expansion (VIII-27) affects the pulse. Fig. VIII-14a shows the situation at  $z = 0$  under the assumption that all the component waves have a zero phase at the pulse centre.

If after some propagation a phase shift  $\varphi(\omega) = \varphi_1(\omega - \omega_0)$  linearly varying with frequency accumulates, the instant at which all the waves are in phase and hence create the pulse is delayed in time (Fig. VIII-14b). The phase shift linear in frequency indeed introduces a time delay (group delay) as concluded before!

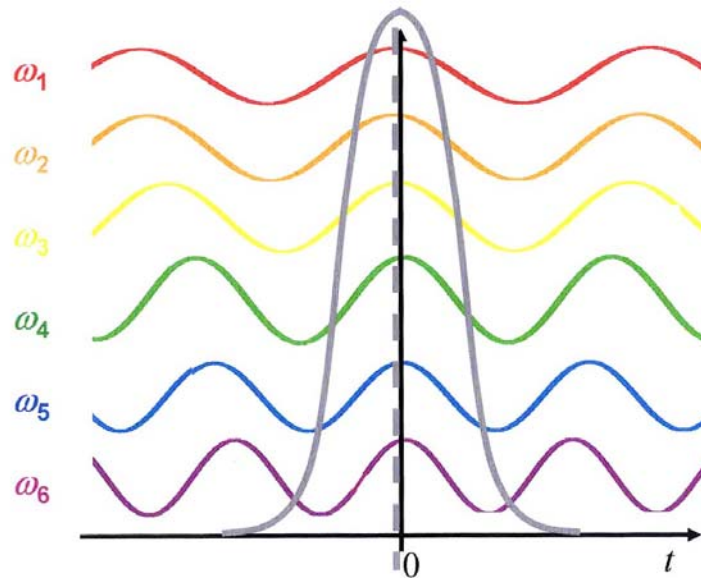


Fig. VIII-14a

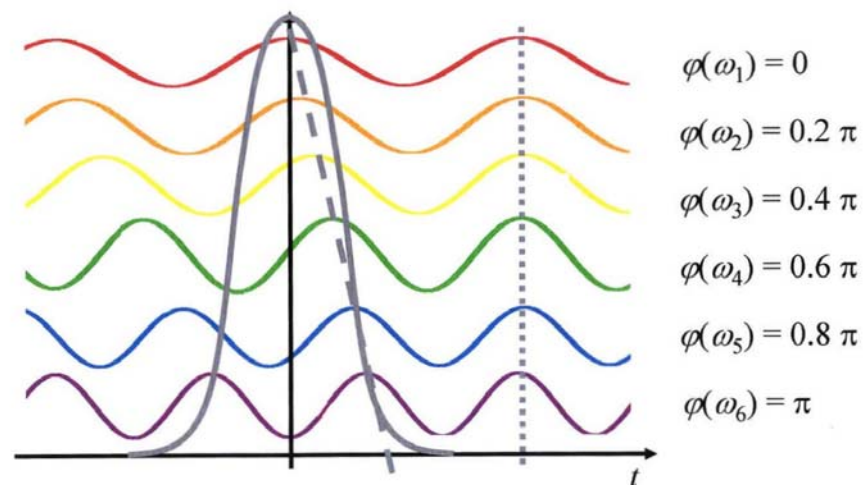


Fig. VIII-14b

### Time-domain description of electromagnetic pulses: decomposition into carrier and envelope

Decomposition of a light wavepacket into carrier and envelope is mainly motivated by the vastly simplified description of its propagation in terms of its complex amplitude  $a(z, t)$ . The respective *pulse propagation equations* applying for different nonlinear and dispersive media and allowing diffraction in the paraxial approximation (3D propagation) can be found in T. Brabec, F. Krausz: *Reviews of Modern Physics* 72, pp. 545-591 (2000).

*How can we define the complex amplitude  $a(t)$  from the waveform  $E(t)$  ?*

The recipe is as simple as follows:

1. calculate or measure the Fourier-spectrum at positive frequencies  $E_+(\omega)$  ;
2. from this determine  $a(\omega)$  by shifting  $E_+(\omega)$  with  $\omega_0$  (see Fig. VIII-12), i.e.  $a(\omega) = E_+(\omega + \omega_0)$
3. calculate  $a(t)$  from  $a(\omega)$  by inverse Fourier transform

How to define  $\omega_0$ ?

In order to avoid fast oscillations in  $a(t)$ , the reference frequency  $\omega_0$  must be close to the centre of  $E_+(\omega)$ . The precise choice is not critical for describing pulse propagation on the basis of  $a(t)$ . There are several possible definitions of  $\omega_0$ , one of which

$$\omega_0 = \bar{\omega} = \frac{\int_0^\infty \omega |E(\omega)|^2 d\omega}{\int_0^\infty |E(\omega)|^2 d\omega} \quad (\text{VIII-37})$$

stands out due to minimising the intensity-weighted phase variation of  $a(t)$ . As a result of this choice, the phase  $\phi(t)$  of  $a(t) = |a(t)|e^{i\phi(t)}$  is a slowly-varying function, i.e. exhibits little variation over the oscillation cycle  $T_0 = 2\pi/\omega_0$ . In these lecture notes we shall use this definition unless otherwise stated.

Is the carrier-envelope decomposition meaningful for pulse durations approaching  $T_0 = 2\pi/\omega_0$ ?

To legitimise the concept of carrier and envelope we must require that  $\omega_0$  and  $a(t)$  remain invariant under a change of  $\phi_0$  in

$$E(t) = \frac{1}{2} a(t) e^{-i\omega_0 t + i\phi_0} + C.C. \quad (\text{VIII-38})$$

A shift of  $\phi_0$  by some  $\Delta\phi$  yields the new wave form  $E'(t) = (1/2)a(t) e^{-i\omega_0 t + i[\phi_0 + \Delta\phi]} + C.C.$  In order that the definition of  $\omega_0$  and  $a(t)$  be self-consistent, the above procedure must yield  $\omega'_0 = \omega_0$ . Note that this requirement is *independent* of the specific definition of  $\omega_0$ . It has been found<sup>4</sup> that this requirement is fulfilled for pulse durations (full width at half maximum, FWHM, of the intensity profile  $|a(t)|^2$ ) approaching the wave cycle  $T_0 = 2\pi/\omega_0$ .

**Lowest-order propagation effect in the time domain: shift of the carrier with respect to the envelope**

When describing pulse propagation in the time domain, which is the method of choice in the presence of nonlinear effects, i.e.  $P_{NL} \neq 0$  in (VIII-19), it is expedient to use the retarded frame of reference defined by (VIII-22), in which the pulse centres at  $\tau = 0$  at any time during propagation.

In this frame we can write

$$E(z, \tau) = \frac{1}{2} a(z, \tau) e^{-i\omega_0 \tau + i\phi'(z)} + C.C. \quad (\text{VIII-39})$$

with

<sup>4</sup> T. Brabec and F. Krausz, *Phys. Rev. Lett.* **78**, 3282 (1997); T. Brabec and F. Krausz, *Rev. Mod. Phys.* **72**, 545-591 (1997).

$$\phi'(z) = \omega_0 \left( \frac{1}{v_{ph}} - \frac{1}{v_g} \right) z \quad (\text{VIII-40})$$

which can be obtained by a simple substitution of the coordinate transformation (VIII-22) into (VIII-18) by assuming  $F(x,y,z) = 1$ . The *complex amplitude*  $a(z, \tau)$  is related to the cycle-averaged intensity (see. IV-34) as

$$I(z, \tau) = \frac{1}{2} \varepsilon_0 c n |a(z, \tau)|^2 \quad (\text{VIII-41})$$

and can be decomposed into a *real amplitude*  $|a(z, \tau)|$  and a phase

$$a(z, \tau) = |a(z, \tau)| e^{i\{\phi_{00} + \phi(z, \tau)\}} \quad (\text{VIII-42})$$

Substituting this expression into (VIII-39) yields

$$\begin{aligned} E(z, \tau) &= \frac{1}{2} a(z, \tau) e^{-i\omega_0 \tau + i\phi'(z)} + c.c. = \\ &= \frac{1}{2} |a(z, \tau)| e^{-i\omega_0 \tau + i\{\phi'(z) + \phi_{00} + \phi(z, \tau)\}} + c.c. \end{aligned} \quad (\text{VIII-43})$$

The phase  $\phi_{00}$  is – by definition – the phase of the electric field at  $z=0$  and at  $\tau=0$ . Its physical meaning is the positioning of the central oscillation cycle in the carrier wave with respect to the pulse peak. The phase term  $\phi'(z)$  shifts the carrier wave as a whole with respect to the pulse peak in proportion to the propagation distance  $z$ . This shift is a consequence of a difference between their propagation velocities (phase and group velocity) in (VIII-40) as shown in Fig. VIII-15.

We can combine  $\phi_{00}$  and  $\phi'(z)$  into a single phase term

$$\phi_0(z) = \phi_{00} + \phi'(z) = \phi_{00} + \omega_0 \left( \frac{1}{v_{ph}} - \frac{1}{v_g} \right) z = \phi_{00} + 2\pi \left( \frac{dn}{d\lambda} \right)_{\lambda_0} z \quad (\text{VIII-44})$$

defining the timing of the central cycle with respect to the envelope at any position during propagation.



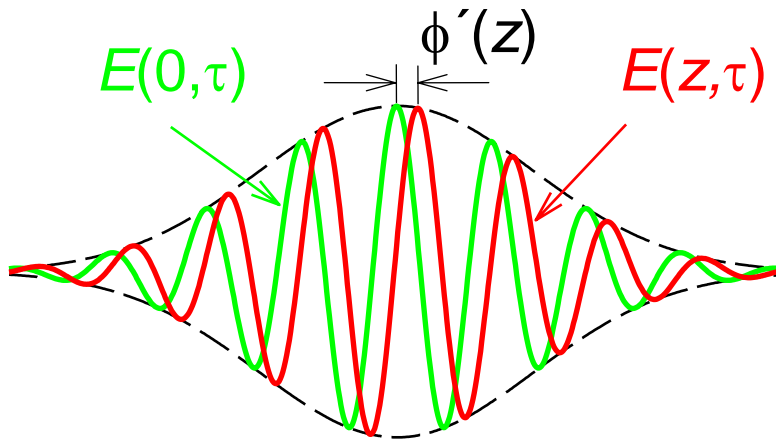


Fig. VIII-15

The last expression in (VIII-44) has been derived from the definition of  $v_{ph}$  and  $v_g$  by substituting  $\omega = 2\pi c/\lambda$  into the relevant expressions (exercise). This expression implies that the carrier is offset with respect to the pulse envelope by a phase shift  $\pi$  over a propagation distance of

$$L_{deph} = \frac{1}{2} \left| \frac{dn}{d\lambda} \right|_{\lambda_0}^{-1} \quad (\text{VIII-45})$$

This characteristic propagation length has been termed *dephasing length*.<sup>5</sup> In the visible and near infrared spectral range it is as short as 10-50 micrometers for transparent materials. The absolute phase typically becomes relevant only if the pulse contains less than three oscillation cycles (Fig. VIII-16).

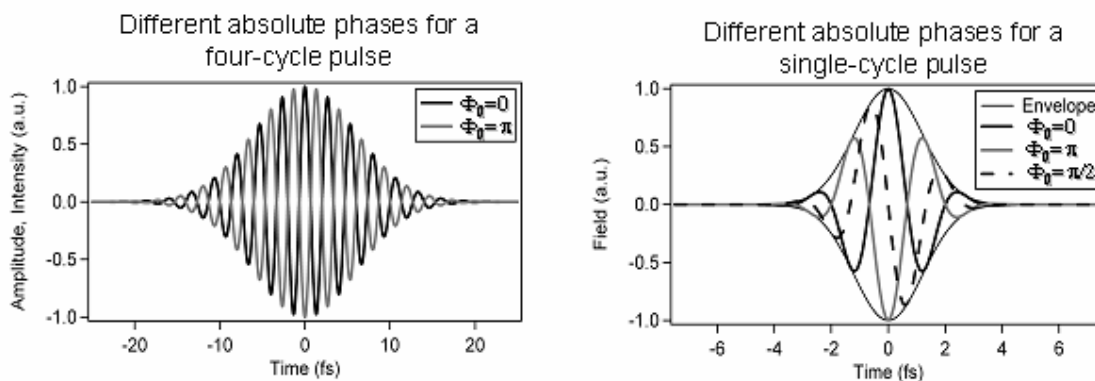


Fig. VIII-16

<sup>5</sup> L. Xu, Ch. Spielmann, A. Poppe, T. Brabec, F. Krausz, and T. W. Hänsch, *Opt. Lett.* 21, 2008 (1996).

### Time-dependent phase: instantaneous frequency, chirped pulse

The time-dependent phase  $\phi(Z, \tau)$  introduces a variation of the instantaneous carrier frequency across the pulse. To see this, consider at some time,  $\tau$ , the total temporal phase of the wave,  $\phi_\tau$

$$\phi_\tau = \omega_0 \tau - \phi(Z, \tau)$$

which contains a rapidly varying term proportional to the carrier frequency and a slowly-varying term  $\phi(\tau)$ . Exactly one period,  $T$ , later, the total phase will – by definition – increase to

$$\phi_{\tau+T} = \phi_\tau + 2\pi = \omega_0[\tau + T] - \phi(Z, \tau + T)$$

where  $\phi(\tau + T)$  is the slowly-varying temporal phase at instant  $\tau + T$ . Subtracting these two equations

$$\phi_{\tau+T} - \phi_\tau = 2\pi = \omega_0 T - [\phi(\tau + T) - \phi(\tau)]$$

and dividing the new one by  $T$ :

$$\frac{2\pi}{T} = \omega_0 - \frac{\phi(\tau + T) - \phi(\tau)}{T}$$

We recognise the left-hand side to be equal to the instantaneous carrier frequency of the wavepacket at the time,  $\tau$ . Further, the second term on the right-hand side can be replaced with  $d\phi/d\tau$  because of the slowly-varying nature of  $\phi(\tau)$ , following from the definition of  $\omega_0$ , Eq. (VIII-38). The *instantaneous carrier frequency* then becomes

$$\omega_{inst}(Z, \tau) = \omega_0 - \frac{\partial\phi(Z, \tau)}{\partial\tau} \quad \text{(VIII-46)}$$

A linear variation of  $\phi(Z, \tau)$  with  $\tau$  merely shifts introduces a constant shift of  $\omega_0$ . If the carrier frequency is defined by (VIII-37), this term generally vanishes. It may, however, emerge, during propagation if, for instance, some nonlinearity causes an asymmetric broadening of the spectrum. The next higher-order term in the Taylor expansion of the temporally-varying phase causes a linear variation of the instantaneous frequency.

### Gaussian pulse modelling of propagation through a linear medium

Well-behaved pulses with a bell-shaped intensity profile can be modelled – in first approximation – as a Gaussian pulse. A Gaussian pulse with a complex Gaussian coefficient  $\gamma = \alpha + i\beta$  can model a pulse with a linear chirp

$$E(\tau) = \frac{1}{2} a_0 e^{-\gamma\tau^2} e^{-i\omega_0\tau} + c.c. = \frac{1}{2} a_0 e^{-\alpha\tau^2} e^{-i(\omega_0\tau + \beta\tau^2)} + c.c. \quad \text{(VIII-47)}$$

Clearly,  $\alpha$  determines the pulse duration  $\tau_p$ . If we define it as the full width at half maximum, FWHM of the intensity profile  $|E(\tau)|^2$  the relationship between  $\tau_p$  and  $\alpha$  reads as

$$\tau_p = \sqrt{\frac{2\ell n^2}{\alpha}} \quad (\text{VIII-48})$$

On the other hand,  $\beta$  results in an instantaneous frequency

$$\omega_{inst}(\tau) = \omega_0 + 2\beta\tau \quad (\text{VIII-49})$$

that increases linearly in time. In analogy to bird sounds, such a pulse is called a "*chirped*" pulse, in this case a linearly-chirped pulse. Fig. VIII-17 shows the electric field of Gaussian pulses for  $\beta > 0$ , positive chirp (a), and for  $\beta < 0$ , negative chirp, (b).

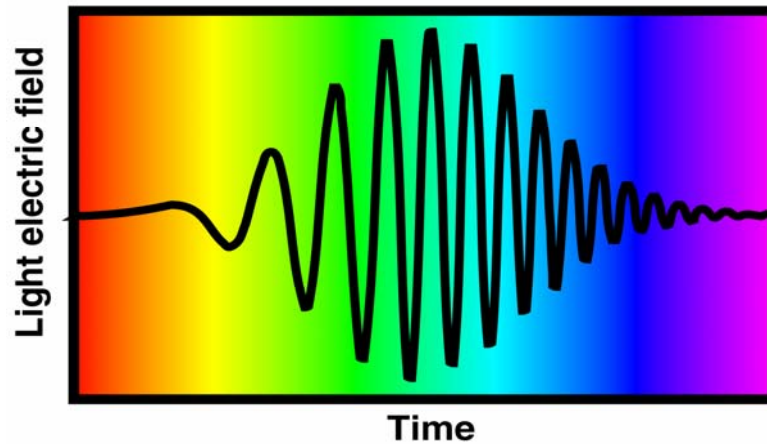


Fig. VIII-17a Positively-chirped Gaussian pulse.

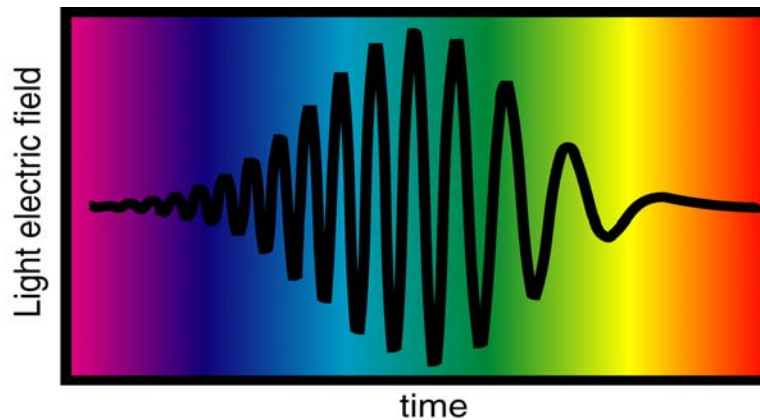


Fig. VIII-17b Negatively-chirped Gaussian pulse.

The frequency spectrum of a Gaussian pulse can be written as (see Chapter IV-4)

$$E_+(\omega) = \frac{1}{2} a_0 \sqrt{\frac{\pi}{\gamma}} e^{-\frac{(\omega-\omega_0)^2}{4\gamma}} = \frac{1}{2} a_0 \sqrt{\frac{\pi}{\alpha + i\beta}} e^{-\frac{(\omega-\omega_0)^2}{4(\alpha + i\beta)}} \quad (\text{VIII-50})$$

The spectrum of a Gaussian pulse is Gaussian too! The frequency bandwidth can now be defined – analogously to  $\tau_p$  – as the full width of half maximum of  $|E(\omega)|^2$

$$\Delta\nu_p = \frac{\Delta\omega_p}{2\pi} = \frac{\sqrt{(2\ell n2)\alpha}}{\pi} \sqrt{1 + \left(\frac{\beta}{\alpha}\right)^2} \quad (\text{VIII-51})$$

Resulting in the *time-bandwidth product*

$$\Delta\nu_p \tau_p = \frac{2\ell n2}{\pi} \sqrt{1 + (\beta/\alpha)^2} = 0.44 \sqrt{1 + \left(\frac{\beta}{\alpha}\right)^2} \geq 0.44 \quad (\text{VIII-52})$$

which is minimum when  $\beta = 0$ . In this case the pulse is referred to as *bandwidth-limited* or *Fourier-limited* because its duration is minimum for the given bandwidth and pulse shape.

Lowest-order pulse shaping effects preserve the shape of a Gaussian pulse

*The power of modelling pulse propagation with a Gaussian pulse relates to the fact that the most important pulse shaping effects such as (lowest-order) amplitude modulation, phase modulation, spectral filtering and dispersion can – in first-order approximation – be modelled to preserve the Gaussian nature of the pulse!*

In fact, multiplying the Gaussian pulse with

$$T_{AM}(\tau) = \exp(-K_{AM}\tau^2) \Rightarrow \alpha' = \alpha + K_{AM} \Rightarrow \text{amplitude modulation} \quad (\text{VIII-53})$$

$K_{AM} > 0$  : pulse shortening  
 $K_{AM} < 0$  : pulse broadening

$$T_{PM}(\tau) = \exp(-iK_{PM}\tau^2) \Rightarrow \beta' = \beta + K_{PM} \Rightarrow \text{phase modulation} \quad (\text{VIII-54})$$

$K_{PM} > 0$  : positive chirp (for  $\beta=0$ )  
 $K_{PM} < 0$  : negative chirp (for  $\beta=0$ )

in the time domain, or with

$$T_{SF}(\omega) = \exp[-K_{SF}(\omega - \omega_0)^2] \Rightarrow \frac{1}{4\gamma'} = \frac{1}{4\gamma} + K_{SF} \Rightarrow \text{spectral filtering} \quad (\text{VIII-55})$$

$K_{SF} > 0$  : spectral narrowing  
 $K_{SF} < 0$  : spectral broadening

$$T_{GDD}(\omega) = \exp[iK_{GDD}(\omega - \omega_0)^2] \Rightarrow \frac{1}{4\gamma'} = \frac{1}{4\gamma} - iK_{GDD} \Rightarrow \text{group-delay dispersion} \quad (\text{VIII-56})$$

performs these manipulations and preserves the pulse as a linearly-chirped Gaussian! As a consequence, basic spectral and temporal manipulations of light pulses can be modelled - in first approximation - in terms of a change in the Gaussian pulse parameter  $\gamma$  analytically!

Phase shift first-order in frequency: a shift in time

$$\phi_1 = \left. \frac{d\phi}{d\omega} \right|_{\omega_0} = -20 \text{ rad fs} \Rightarrow T_g = -20 \text{ fs}$$

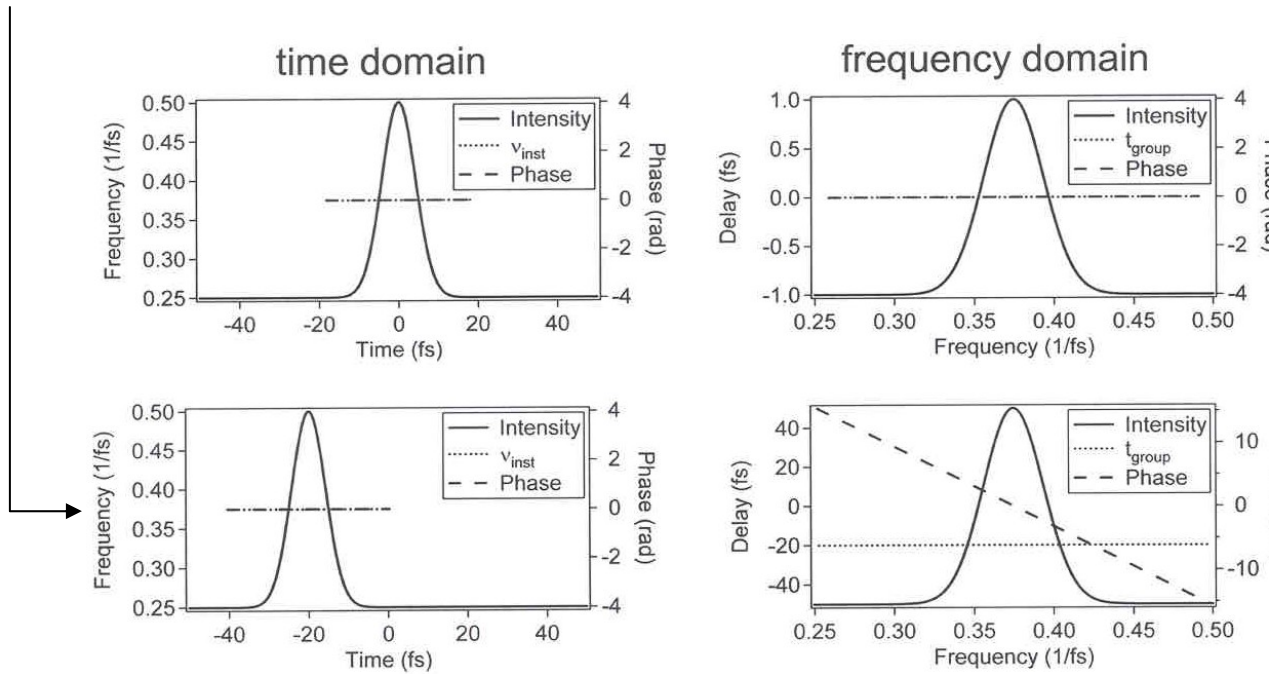


Fig. VIII-18

First-order phase in time: a frequency shift

$$\phi_1 = \left. \frac{\partial\phi}{\partial\tau} \right|_0 = -0.07 \times 2\pi \text{ rad/fs} \Rightarrow \Delta\nu = -0.07 \text{ fs}^{-1}$$

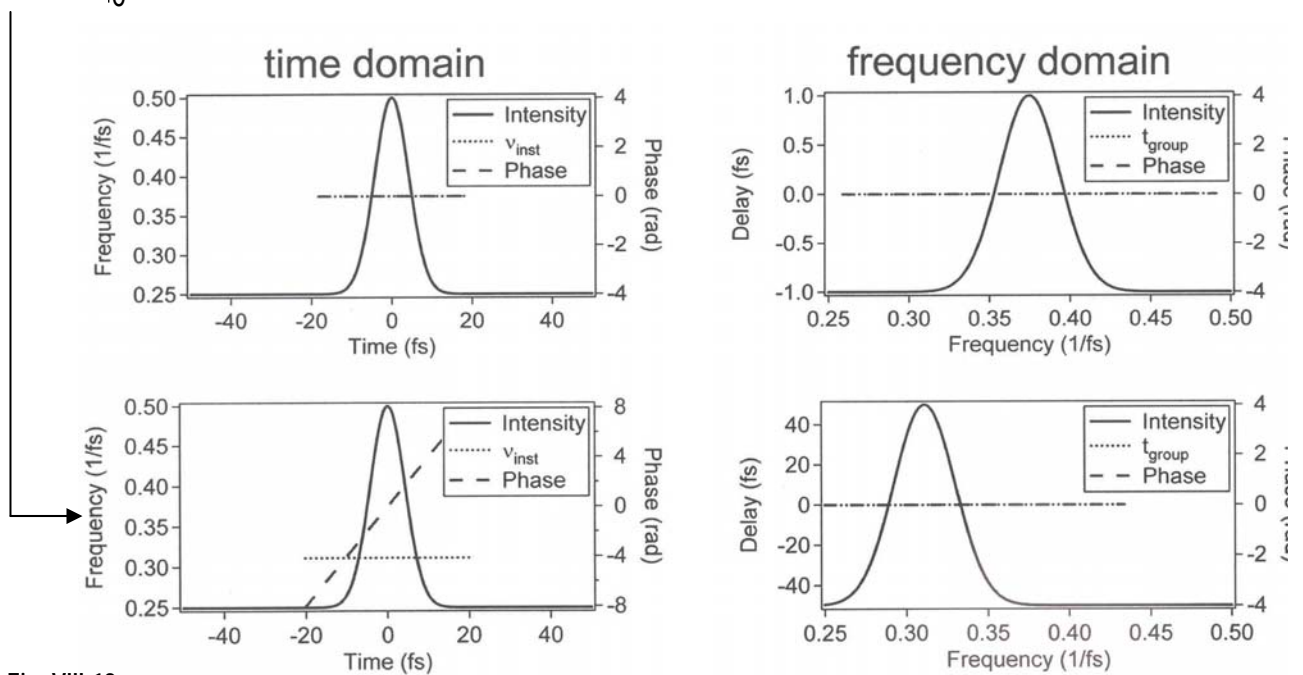


Fig. VIII-19

Phase shift second-order in frequency: positive or negative linear chirp

$$\varphi_2 = \left. \frac{d^2\varphi}{d\omega^2} \right|_{\omega_0} = 290 \text{ rad fs}^2 \Rightarrow \beta = -\left. \frac{1}{2} \frac{\partial^2\phi}{\partial\tau^2} \right|_0 = 0.016 \text{ rad/fs}^2$$

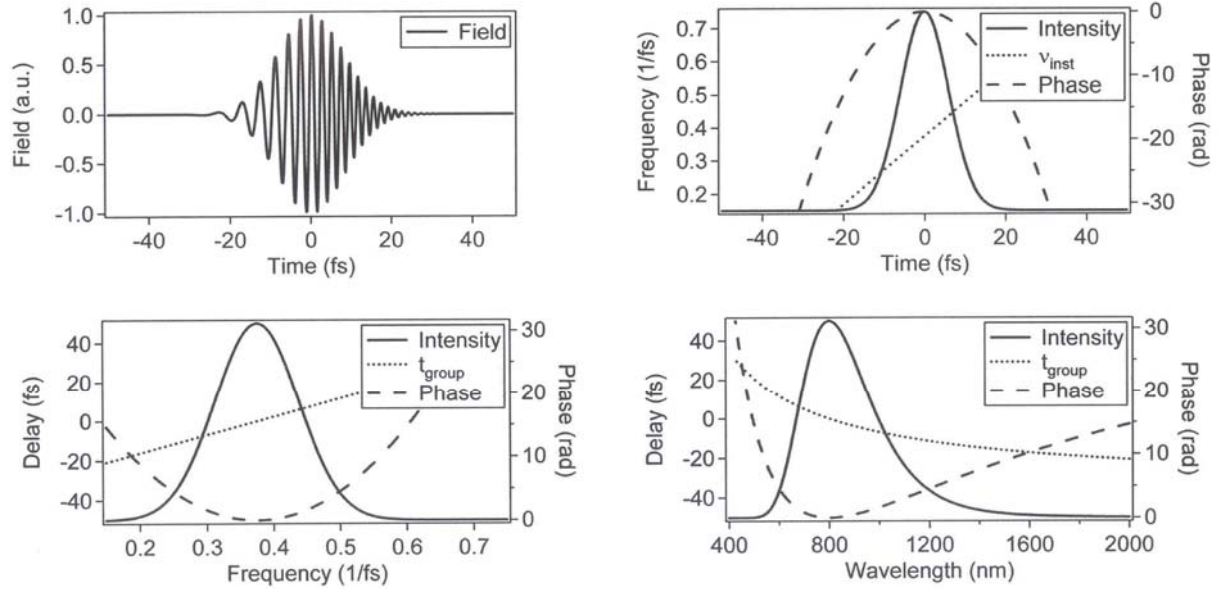


Fig. VIII-20

$$\varphi_2 = \left. \frac{d^2\varphi}{d\omega^2} \right|_{\omega_0} = -290 \text{ rad fs}^2 \Rightarrow \beta = -\left. \frac{1}{2} \frac{\partial^2\phi}{\partial\tau^2} \right|_0 = -0.016 \text{ rad/fs}^2$$

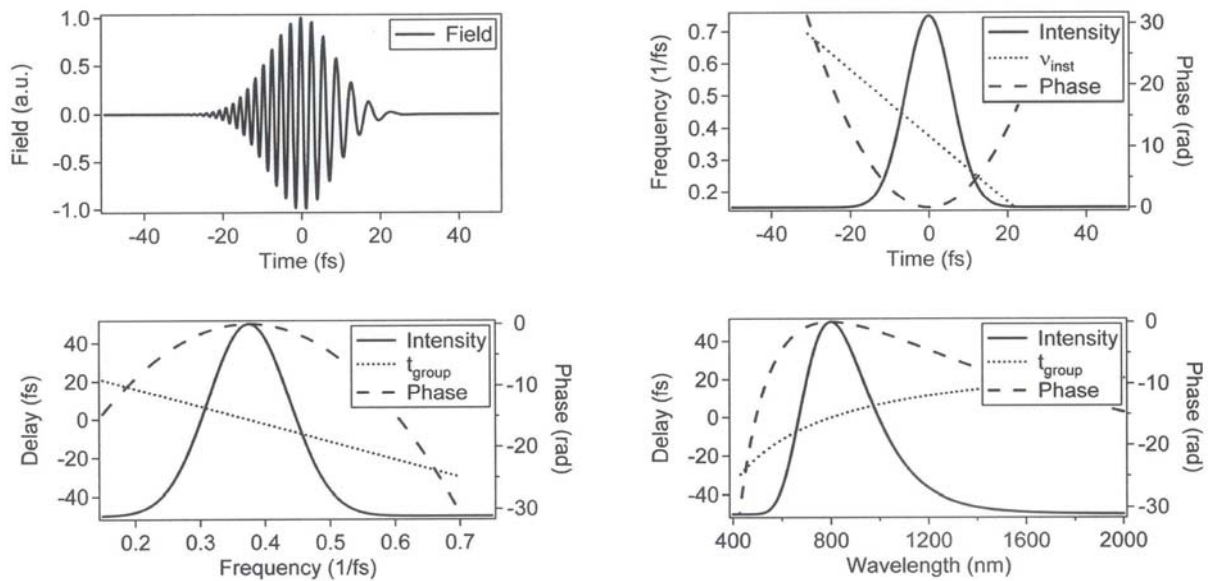


Fig. VIII-21

Beyond imposing a linear chirp, the quadratic spectral phase shift has stretched a pulse with originally 3-fs duration to a ~14-fs pulse.

Phase shift third-order in frequency: satellite pulses

$$\varphi_3 = \left. \frac{d^3\varphi}{d\omega^3} \right|_{\omega_0} = 3 \times 10^4 \text{ rad fs}^3$$

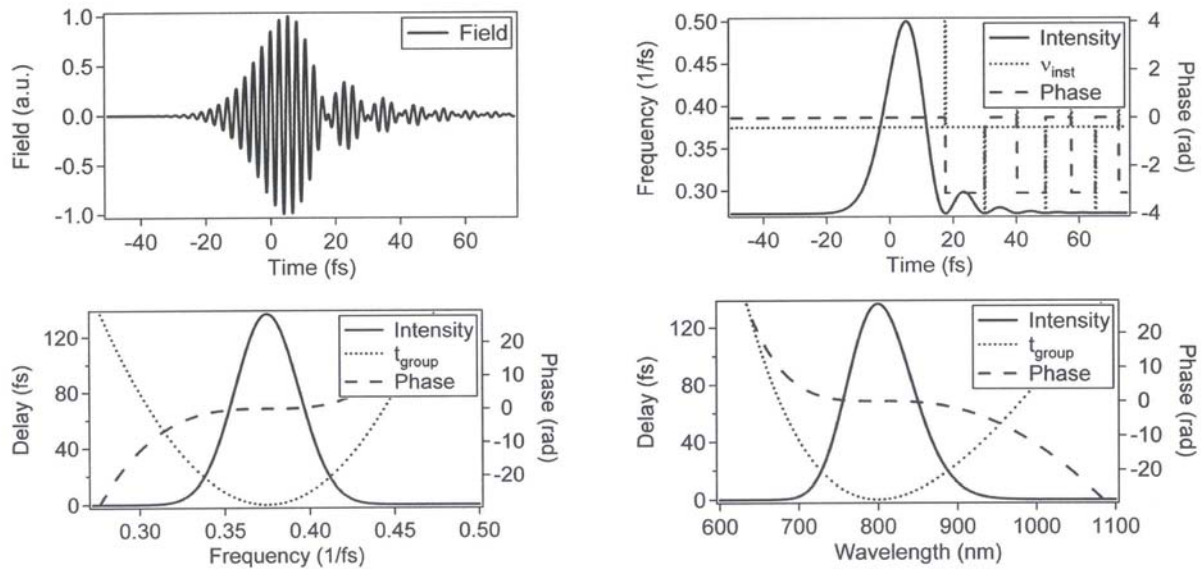


Fig. VIII-22 Trailing satellite pulses are indicative of positive spectral cubic phase.

$$\varphi_3 = \left. \frac{d^3\varphi}{d\omega^3} \right|_{\omega_0} = -3 \times 10^4 \text{ rad fs}^3$$

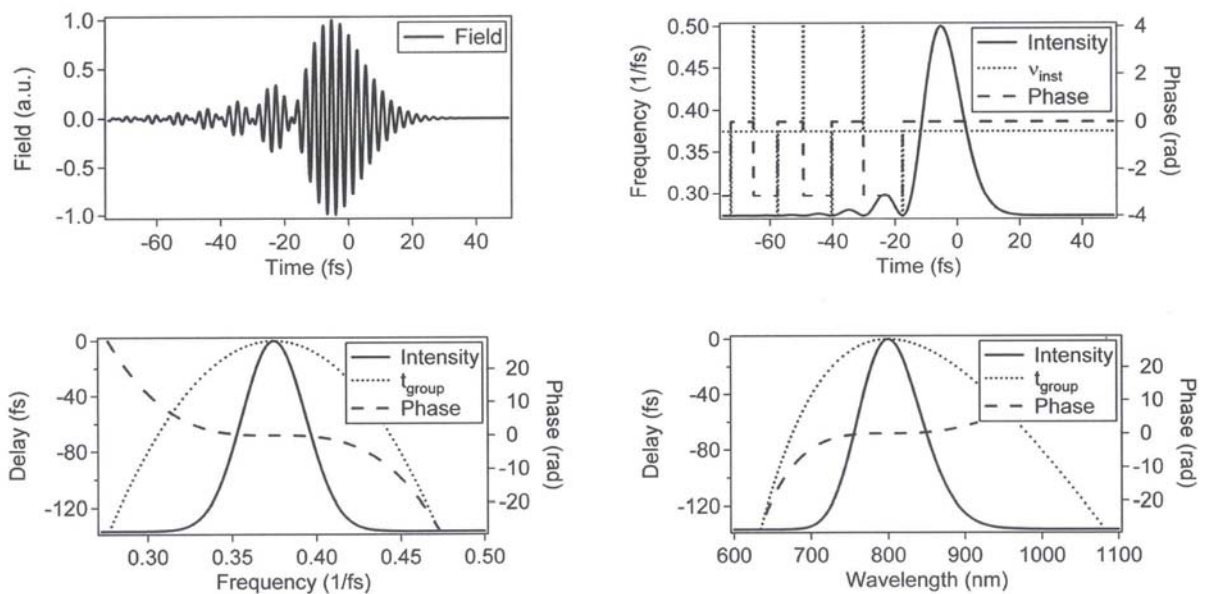


Fig. VIII-23 Leading satellite pulses are indicative of positive spectral cubic phase.

Phase shift fourth-order in frequency: leading and trailing pulse wings

$$\Phi_4 = \left. \frac{d^4 \Phi}{d\omega^4} \right|_{\omega_0} = 4 \times 10^5 \text{ rad fs}^4$$

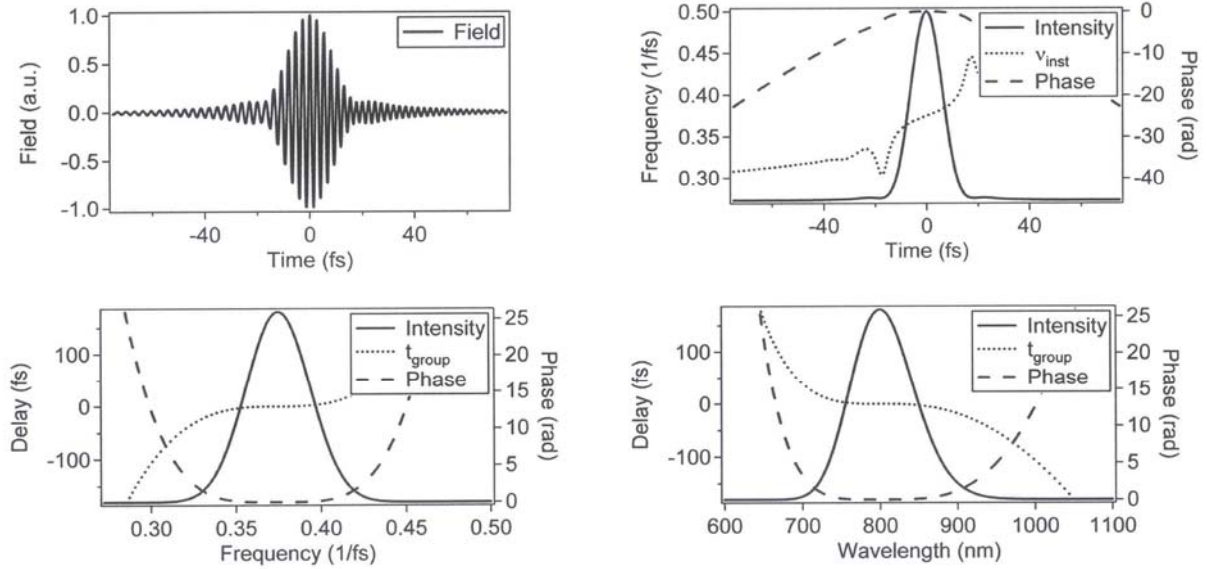


Fig. VIII-24 Positive quadratic spectral phase implies higher frequencies in the trailing wing.

$$\Phi_4 = \left. \frac{d^4 \Phi}{d\omega^4} \right|_{\omega_0} = -4 \times 10^5 \text{ rad fs}^4$$

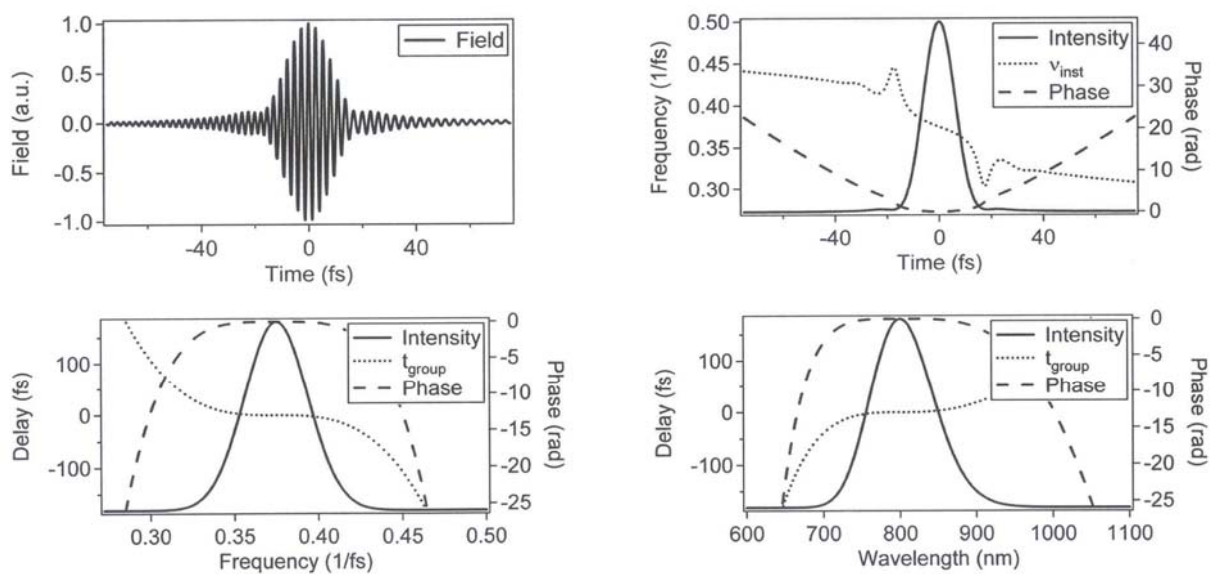


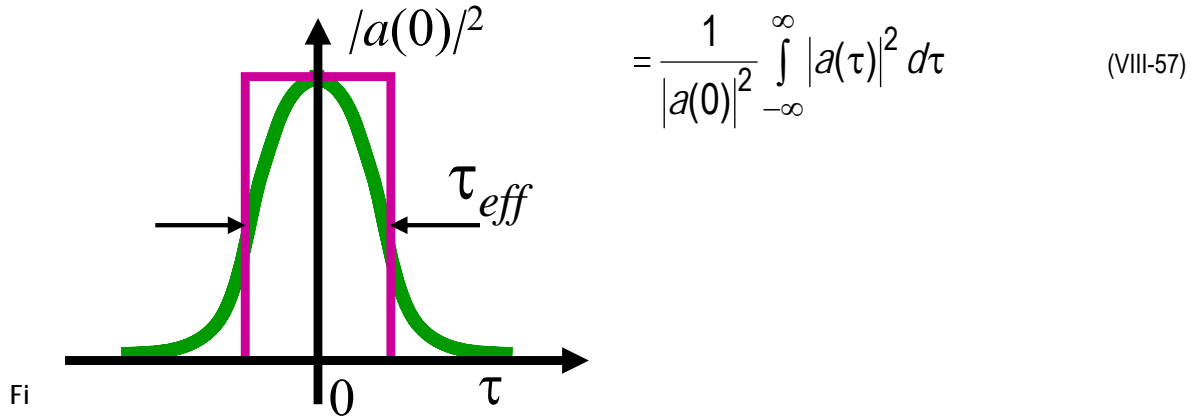
Fig. VIII-25 Negative quadratic spectral phase implies higher frequencies in the leading wing.



Pulse characteristics: duration, time-bandwidth product

There are many possible definitions of the duration (or “width” or “length”) of a wavepacket.

*Effective width:* the width of a rectangle whose height and area are the same as those of the pulse (Fig. VIII-26).



*Rms (root-mean-squared) width:* second-order moment of the intensity profile (Fig. VIII-27).

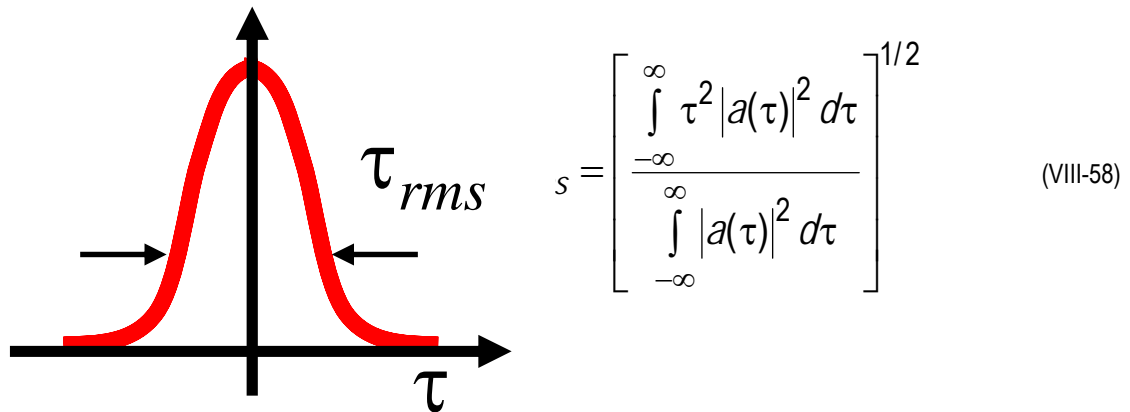


Fig. VIII-27

*FWHM (full width at half maximum):* distance between the half-maximum points (Fig. VIII-28).

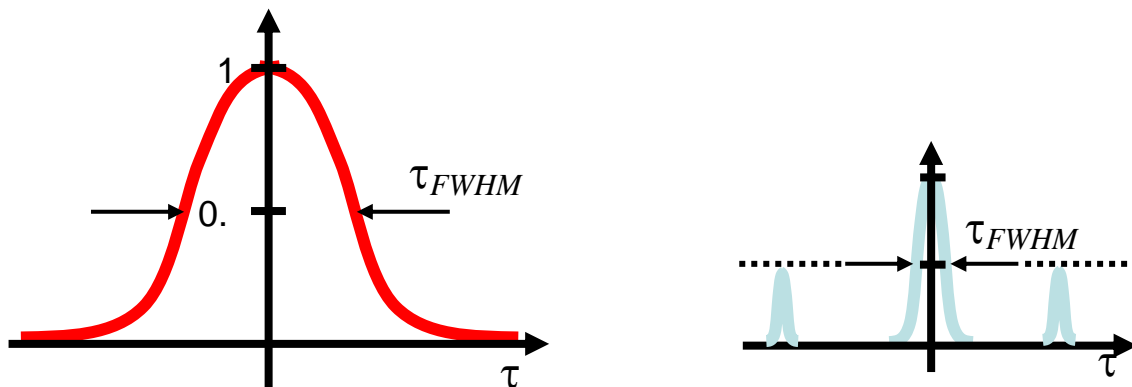


Fig. VIII-28

The FWHM is the most frequently used measure for the duration of ultrashort light pulses. However, it may be used only for well-behaved pulses with a bell-shaped profile and small wings or satellites because the latter are ignored up to a height of 49.99%!

### Time-bandwidth product

According to the Fourier theorem the product of a function's extension in the time domain,  $\Delta\tau$ , and the frequency domain,  $\Delta\omega$ , has a minimum.

Let us define the widths of the intensity profile  $f(\tau)$  and its Fourier-transform,  $F(\omega)$ , using the definition (VIII-57) and assuming  $f(\tau)$  and  $F(\omega)$  peak at 0:

$$\Delta\tau = \frac{1}{f(0)} \int_{-\infty}^{\infty} |f(\tau)| d\tau \quad ; \quad \Delta\omega = \frac{1}{F(0)} \int_{-\infty}^{\infty} |F(\omega)| d\omega$$

Then

$$\Delta\tau \geq \frac{1}{f(0)} \int_{-\infty}^{\infty} f(\tau) d\tau = \frac{1}{f(0)} \int_{-\infty}^{\infty} f(\tau) e^{i0\tau} d\tau = \frac{F(0)}{f(0)}$$

and

$$\Delta\omega \geq \frac{1}{F(0)} \int_{-\infty}^{\infty} F(\omega) d\omega = \frac{1}{F(0)} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega[0]} d\omega = \frac{2\pi f(0)}{F(0)}$$

Combining the two latter results

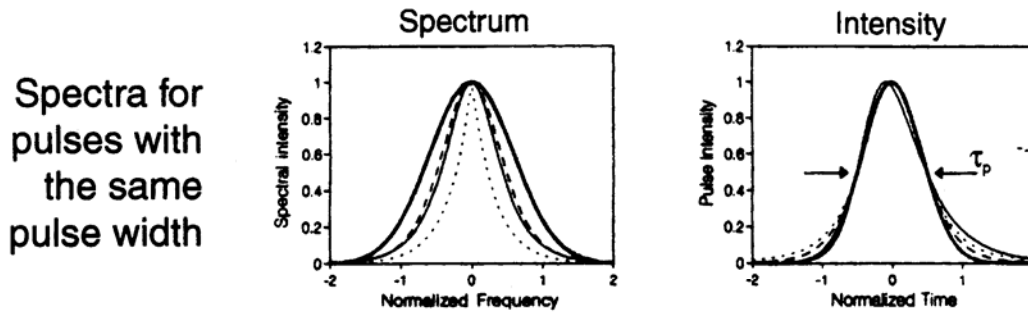
$$\Delta\omega\Delta\tau \geq 2\pi \frac{f(0)}{F(0)} \frac{F(0)}{f(0)} \Rightarrow \Delta\omega\Delta\tau \geq 2\pi \quad ; \quad \Delta\nu\Delta\tau \geq 1 \quad (\text{VIII-59})$$

Different definitions of the widths yield different constants, which – for most definitions (incl. FWHM) also depend on the pulse shape.

Temporal and spectral shapes of simple electromagnetic pulses<sup>6</sup>

### Common pulse envelopes (with $\tau_p = \text{Intensity FWHM}$ ):

- Gaussian pulse       $\mathcal{E}(t) \propto \exp[-1.385(t/\tau_p)^2]$
- - - - - sech - pulse       $\mathcal{E}(t) \propto \text{sech}[1.763(t/\tau_p)]$
- ..... Lorentzian pulse       $\mathcal{E}(t) \propto [1 + 1.656(t/\tau_p)^2]^{-1}$
- asymm. sech pulse       $\mathcal{E}(t) \propto [\exp(t/\tau_p) + \exp(-3t/\tau_p)]^{-1}$



Field envelope	Intensity profile	$\tau_p$ (FWHM)	Spectral profile	$\Delta\omega_p$ (FWHM)	TBP
Gauss	$e^{-2(t/\tau_G)^2}$	$1.177\tau_G$	$e^{-(\omega\tau_G)^2/2}$	$2.355/\tau_G$	0.441
sech	$\text{sech}^2(t/\tau_s)$	$1.763\tau_s$	$\text{sech}^2(\pi\omega\tau_s/2)$	$1.122/\tau_s$	0.315
Lorentz	$[1 + (t/\tau_L)^2]^{-2}$	$1.287\tau_L$	$e^{-2 \omega \tau_L}$	$0.693/\tau_L$	0.142
asymm. sech	$[e^{t/\tau_a} + e^{-3t/\tau_a}]^{-2}$	$1.043\tau_a$	$\text{sech}(\pi\omega\tau_a/2)$	$1.677/\tau_a$	0.278
rectang.	1 for $ t/\tau_r  \leq \frac{1}{2}$ , 0 else	$\tau_r$	$\text{sinc}^2(\omega\tau_r)$	$2.78/\tau_r$	0.443

<sup>6</sup> J.-C. Diels, W. Rudolph, *Ultrashort Laser Pulse Phenomena* (Academic Press, San Diego, USA, 1996).

## Transformation between wavelength and frequency

The spectrum and spectral phase is often measured versus wavelength  $S_\lambda(\lambda) \equiv |E_\lambda(\lambda)|^2$ ,  $\varphi_\lambda(\lambda)$ , rather than frequency,  $S_\omega(\omega) \equiv |E_\omega(\omega)|^2 \equiv |E(\omega)|^2$ ,  $\varphi_\omega(\omega) \equiv \varphi(\omega)$ , therefore a transformation rule is required.

By using  $\omega = 2\pi c/\lambda$

The phase can be easily transformed

$$\varphi_\lambda(\lambda) = \varphi_\omega(2\pi c/\lambda) \quad (\text{VIII-60a})$$

For transforming the spectrum, we note that  $S_\lambda(\lambda)$  and  $S_\omega(\omega)$  are the energy density within a unit wavelength and frequency interval, hence the total energy in the pulse is given by

$$W_p = \int_{-\infty}^{\infty} S_\lambda(\lambda) d\lambda = \int_{-\infty}^{\infty} S_\omega(\omega) d\omega$$

Changing variables and using  $d\omega/d\lambda = -2\pi c/\lambda^2$

$$\int_{-\infty}^{\infty} S_\omega(\omega) d\omega = \int_{\infty}^{-\infty} S_\omega(2\pi c/\lambda) \frac{-2\pi c}{\lambda^2} d\lambda = \int_{-\infty}^{\infty} S_\omega(2\pi c/\lambda) \frac{2\pi c}{\lambda^2} d\lambda$$

From which we obtain

$$S_\lambda(\lambda) = S_\omega(2\pi c/\lambda) \frac{2\pi c}{\lambda^2} \quad (\text{VIII-60b})$$

Eq. (VIII-60) implies different spectral shapes of broadband signals vs. frequency and wavelength.

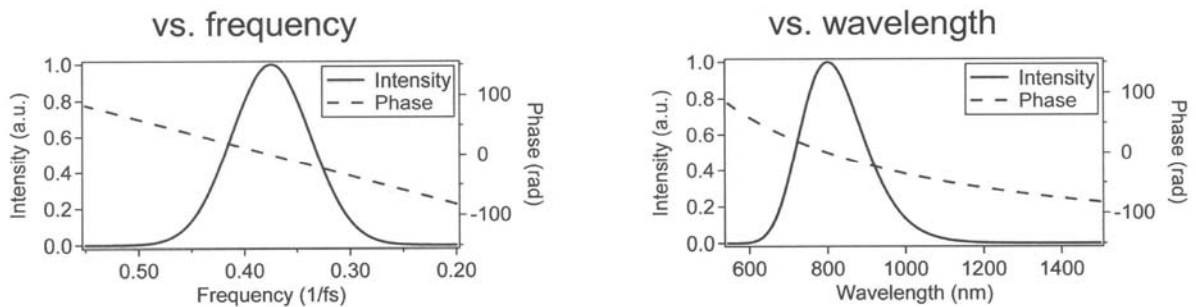


Fig. VIII-29 The spectral shape of a broadband pulse is different vs. frequency and wavelength.

Bandwidth expressed in units of frequency, wave number and wavelength

Wave number:  $\frac{1}{\lambda}$

$$v = \frac{c}{\lambda} \Rightarrow dv = d\left(\frac{1}{\lambda}\right)c \Rightarrow \boxed{\delta\left(\frac{1}{\lambda}\right) = \delta v \frac{1}{c}} \quad (\text{VIII-61})$$

Wavelength:  $\lambda$

$$v = \frac{c}{\lambda} \Rightarrow \left| \frac{dv}{d\lambda} \right| = \frac{c}{\lambda^2} \Rightarrow \boxed{\delta\lambda = \delta v \frac{\lambda^2}{c}} \quad (\text{VIII-62})$$

In units of frequency and wave number, the bandwidth of a Fourier-limited pulse is independent of its carrier frequency. In wavelength units, the bandwidth depends on the carrier wavelength, too.

### Phase wrapping and unwrapping

The phase is often represented within the interval  $(-\pi, \pi)$  and called “wrapped” phase. It is often helpful to avoid sudden phase jumps (e.g. if it has to be differentiated), which frequently show up in this representation, by “unwrapping” it. This can be done by adding or subtracting  $2\pi$  whenever there is a  $2\pi$  phase jump (Fig. VIII-30).

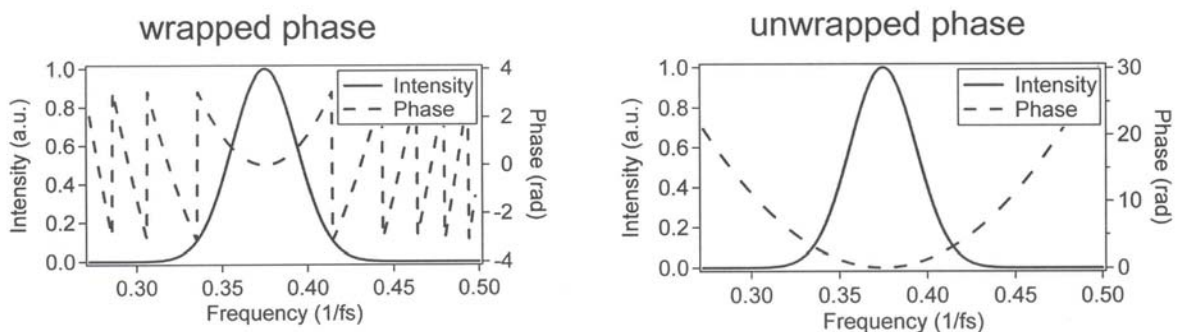


Fig. VIII-30 Phase unwrapping for a pulse with a quadratic spectral phase.

### Phase blanking

In calculations or in evaluation of measured data, we often end up with dramatic phase variations in spectral or temporal ranges, where the intensity is (close to) zero (Fig. VIII-31, left panel). When the intensity is zero, the phase is irrelevant. When the intensity is nearly zero, the phase is nearly irrelevant.

Phase blanking involves simply not plotting the phase when the intensity is close to zero (Fig. VIII-31, right panel). One has to choose the intensity level, below which the phase is disregarded, very carefully.

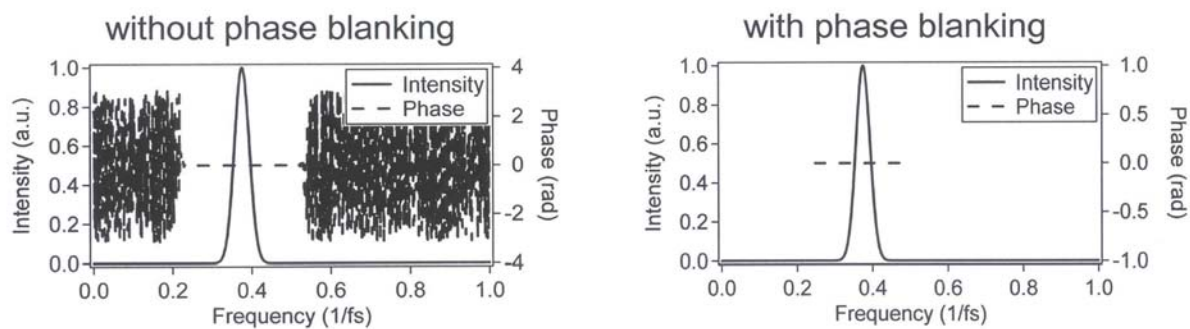


Fig. VIII-31