VII. Relativistic optics

Electromagnetic fields in inertial frames of reference

Galilean transformation

Before 1900 the space and time coordinates (*x, y, z, t*) and (*x', y', z', t'*) of reference frames *K* and *K'* moving with constant speeds differing by ***v*** were believed to be related by Galilean transformation

$$\begin{aligned} \mathbf{r}' &= \mathbf{r} - \mathbf{v}t \\ t' &= t \end{aligned} \qquad \text{(VII-1)}$$

provided the origins in space and time are chosen suitably. Both the Newton equation of classical mechanics and the Schrödinger equation of quantum mechanics can be shown to be invariant under Galilean transformation. For instance, if in *K'* the Newton equations for a mechanical system consisting of a group of particles interacting via two-body potentials read as

$$m_i \frac{d^2\mathbf{r}_i'}{dt'^2} = -\nabla_i' \sum_j V_{ij}(\mathbf{r}_i' - \mathbf{r}_j') \qquad \text{(VII-2)}$$

then in *K* the equation of motion has the same form

$$m_i \frac{d^2\mathbf{r}_i}{dt^2} = -\nabla_i \sum_j V_{ij}(\mathbf{r}_i - \mathbf{r}_j) \qquad \text{(VII-3)}$$

By contrast, the equation governing wave phenomena is not preserved under the transformation (VII-1). For a field $\psi(\mathbf{r}', t')$ satisfying the wave equation

$$\nabla'^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2} = 0 \qquad \text{(VII-4)}$$

in *K'* the Galilean transformation yields the equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{2}{c^2}\mathbf{v}\cdot\nabla\frac{\partial\psi}{\partial t} - \frac{1}{c^2}\mathbf{v}\cdot\nabla\left(\mathbf{v}\cdot\nabla\psi\right) = 0 \qquad \text{(VII-5)}$$

in the reference frame *K*. For sound waves, the lack of invariance under transformation from one frame moving with a constant velocity to another is acceptable given the fact that propagation of these waves relies on a transmitting medium. The preferred reference frame *K'* in which (VII-4) is valid is obviously the frame in which the propagation medium is at rest.

Postulates of Einstein's special theory of relativity

Analogously, a preferred reference frame for light propagation calls for a medium through which light can propagate. This medium has been called the *ether* and assumed to permeate all space and to be of negligible density and to have negligible interaction with matter. However, efforts to observe the motion of laboratories on the Earth relative to the rest frame of ether, for example, the famous Michelson-Morley experiment, had failed. Fizeau's experiments on the velocity of light in moving fluids could also be understood in terms of the ether hypothesis only by assuming that the ether was dragged along partially by the moving fluid, with the effectiveness of the medium in dragging the ether related to its index of refraction!

These experiments made Einstein abandon the hypothesis of an ether. The only alternative was to modify the transformation of the space and time coordinates between uniformly moving reference frames so that Maxwell's equations become invariant under the new transformation. This immediately implied that the laws of mechanics were in need of modifications.

To establish the new transformation, Einstein introduced two postulates.

1. Postulate of relativity: The laws of nature and the results of all experiments performed in a uniformly moving frame of reference are independent of the translational motion of the system as a whole.

   These equivalent coordinate systems are called *inertial* reference frames.

2. Postulate of the constancy of the speed of light: The speed of light is constant in every frame of reference, independently of the motion of its source.

From these two postulates the rules of the new transformation of space and time, named after Lorentz, can be derived in a straightforward manner[1]. Before we address the derivation of the Lorentz transformation of coordinates, we summarize a few major experiments backing these postulates, performed long after Einstein proposed them.

Experimental verification of Einstein's relativity

It is a general belief that the null result of the Michelson-Morles experiment (1887) provided an unambiguous evidence for the second postulate of the Special Theory of Relativity. This is not true. This experiment can also be explained without abandoning the concept of an ether by the hypothesis of the FitzGerald-Lorentz contraction. Compelling evidence came long after the formulation of the theory by Einstein.[2] One of the most beautiful class of experiments draws on gamma-ray emission from nuclei with an extremely narrow bandwidth resulting from the Mössbauer effect.

Let's suppose that the inertial frames K and K' are connected with the Galilean coordinate transformation. The phase of a plane wave must be invariant quantity, the same in all coordinate frames:

$$\phi = \omega\left(t - \frac{\mathbf{n}\cdot\mathbf{r}}{c}\right) = \omega'\left(t' - \frac{\mathbf{n}'\cdot\mathbf{r}'}{c'}\right) \tag{VII-6}$$

Expressing t and $\mathbf{r}$ in terms of t' and $\mathbf{r}'$ from (VII-1) leads to

$$\omega t'\left(1 - \frac{\mathbf{n}\cdot\mathbf{v}}{c}\right) - \omega\,\frac{\mathbf{n}\cdot\mathbf{r}'}{c} = \omega' t' - \omega'\frac{\mathbf{n}'\cdot\mathbf{r}'}{c'} \tag{VII-7}$$

[1] The general structure of the Lorentz transformation can be deduced from the first postulate alone, see e.g. N. D. Mermin, Relativity without light, *Am. J. Phys.* 52, 119 (1984).
[2] A summary of available evidence is given by Shankland *et al., Rev. Mod. Phys.* 27, 167 (1955).

This equality must hold for values of *t'* and **r'**. As a consequence, the coefficients of *t'*, x' , y' , z' on both sides must be equal. From this requirement, we obtain the Doppler-shift formulas for Galilean relativity:

$$\omega' = \omega(1 - \frac{\mathbf{n}\cdot\mathbf{v}}{c}) \qquad \text{(VII-8a)}$$

$$c' = c - \mathbf{n}\cdot\mathbf{v} \qquad \text{(VII-8b)}$$

$$\mathbf{n}' = \mathbf{n} \qquad \text{(VII-8c)}$$

The direction of the wave vector (given by **n**) appears to be invariant. However, the direction of energy flow changes from one frame to the other. Suppose that inertial frame *K* is the preferential reference frame, in which the ether is at rest, hence the light wave propagates at *c* with both the wave vector and the direction of energy flow (direction of movement of the wave packet represented by the segments of a plane wave in Fig. VII-1) directed along **n**.

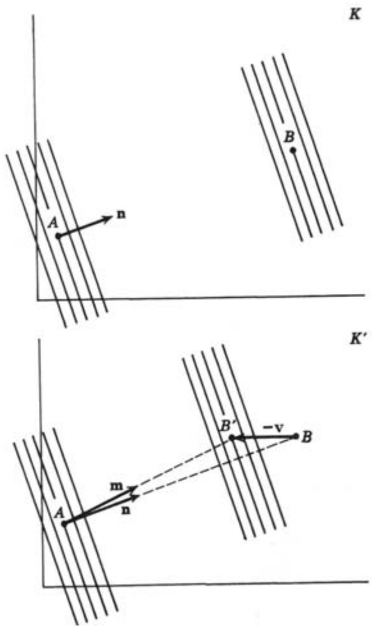

Fig. VII-1

The direction of energy flow is not parallel to **n** in *K'* but parallel with the unit vector

$$\mathbf{m} = \frac{c\mathbf{n} - \mathbf{v}}{|c\mathbf{n} - \mathbf{v}|} \qquad \text{(VII-9)}$$

The experiments are performed in the laboratory, therefore it is expedient to be able to express the Doppler formulas (VII-8) in terms of **m** appropriate to the laboratory rather than **n**. To this end, we write **n** in terms of **m**. In the limit of $|v_0 / c| << 1$ we can – according to Fig. VII-2 – approximately write

$$\mathbf{n} \approx \left(1 - \frac{\mathbf{m}\cdot\mathbf{v}_0}{c}\right)\mathbf{m} + \frac{\mathbf{v}_0}{c} \qquad \text{(VII-10)}$$

where v_0 is the velocity of the laboratory relative to the ether rest frame.

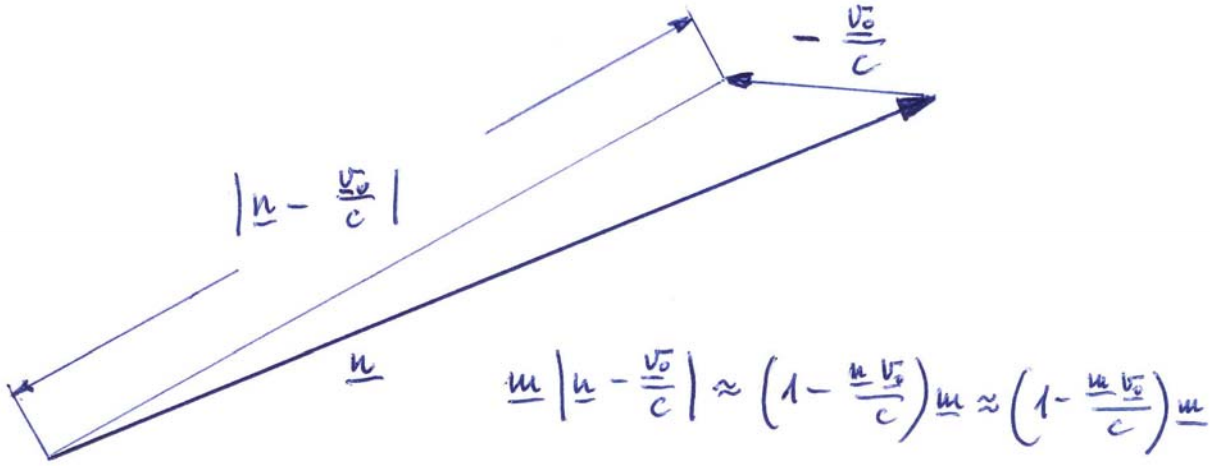

Fig. VII-2

A plane wave having a frequency ω in the ether rest frame will have a frequency

$$\omega_0 = \omega\left(1-\frac{\mathrm{n}\cdot\mathrm{v}_0}{c}\right) \tag{VII-11a}$$

in the laboratory frame and a frequency

$$\omega_1 = \omega\left(1-\frac{\mathrm{n}\cdot\mathrm{v}_1}{c}\right) \tag{VII-11b}$$

in a frame K_1 moving with a velocity $\mathrm{v}_1 = \mathrm{u}_1 + \mathrm{v}_0$ relative to the ether rest frame.
The frequencies ω_1 and ω_0 are connected by

$$\omega_1 \approx \omega_0\left[1-\frac{\mathrm{u}_1}{c}\left(\mathrm{m}+\frac{\mathrm{v}_0}{c}\right)\right] \tag{VII-12}$$

We obtained (VII-12) from (VII-11) by eliminating n, which can not be measured, by means of (VII-10). Equation (VII-12) is a consequence of the assumed validity of the wave equation (VII-4) in the ether rest frame and that of Galilean coordinate transformation connecting inertial frames. Owing to the appearance of v_0 it is obviously able to prove or disprove the existence of a preferred frame of reference and the validity of Galilean relativity.

Differences between ω_1 and ω_0 can be measured accurately if the source and detectors have a narrow bandwidth. These requirements can be best met by using two Mössbauer systems of identical ω_0, one emitter and one absorber. Suppose they move with u_1 and u_2 in the laboratory. Eq. (VII-12) then implies for the difference frequency between the emitter and the absorber

$$\frac{\omega_1-\omega_2}{\omega_0} = \frac{1}{c}\left(\mathrm{u}_1-\mathrm{u}_2\right)\left(\mathrm{m}+\frac{\mathrm{v}_0}{c}\right) \tag{VII-13}$$

In the first resonant absorption experiment of this type[3] the emitter and the absorber were located on the opposite ends of a rod of length 2*R* rotated about its centre with an angular velocity Ω as depicted in Fig. VII-3. For this specific case (u_1 - u_2)m= 0 and the fractional frequency difference is predicted as

$$\frac{\omega_1-\omega_2}{\omega_0} = \frac{2\Omega R}{c^2}\left|(\mathrm{v}_0)_\perp\right| \sin \Omega t \tag{VII-14}$$

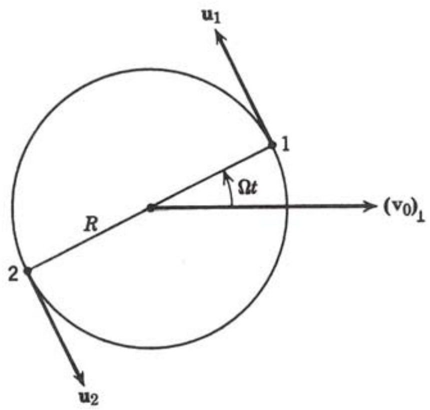

Fig. VII-3

In the experiment of Champeney, Isaak and Khan the Mössbauer line was at a photon energy of 14.4 keV with a fractional bandwidth of 2x10^{-12}, the emitter and the absorber foils were separated by 2*R* = 8cm and the highest rotation speed close to 8000 s^{-1}. At a rotational speed of ~6000 s^{-1} Eq. (VII-14) yields a difference frequency equal to the Mössbauer line width for an ether drift velocity of ~200m/s.
Champeney and co performed the measurement in 4-hour cycles and searched for different ether drift velocities relative to their laboratory due to the earth's rotation. Their experiment yielded a maximum ether drift speed component orthogonal to the earth's axis of rotation of < 5 m/s. An improved experiment in 1970 set the limit of 5 cm/s. *Clearly, the existence of a preferential reference frame and with that the idea of the ether must be abandoned, the first postulate of the special theory of relativity has been confirmed.*

The second postulate, the constancy of the speed of light irrespective of the motion of its source could be unquestionably verified also only many years after postulating this law, in a beautiful experiment[4] performed at CERN, Geneva, in 1964. The speed of 6 GeV gamma-ray photons produced in the decay of energetic pions flying with a speed of v = 0.99975c was measured by time of flight. Within the experimental error the speed c' of the photons emitted by the fast moving particles was found to be equal to c, written as $c' = c + kv$, the experiment yielded $k = (0\pm1.3)\mathrm{x}10^{-4}$. Other experiments[5] confirmed this result and provided evidence for the constancy of the light speed over its frequency, up to photon energies of 7 GeV[6], *establishing conclusively the validity of the second postulate of special relativity.*

Clearly, the constancy of the speed of light, independently of the motion of the source, destroys the concept of time as a universal variable independent of the spatial coordinates. Experiments taught us that light propagates in the

[3] D. C. Champeney, G. R. Isaak, A. M. Khan, *Phys. Lett.* 7, 241 (1963); G. R. Isaak, *Phys. Bull.* 21, 255 (1970).
[4] T. Alvager, J. M. Bailey, F. J. M. Farley, J. Kjellman, and I. Wallin, *Phys. Lett.* 12, 260 (1964).
[5] G. R. Kalbfleisch, N. Baggett, E. C. Fowler, *Phys. Rev. Lett.* 43, 1361 (1979).
[6] B. C. Brown *et al. Phys. Rev. Lett.* 30, 763 (1973).

same way in any inertial frame, consequently Maxwell's equations must be invariant under the transformation connecting two inertial frames. Galilean coordinate transformation must be modified to meet this requirement.

Lorentz transformation – derivation from the requirement of the relativistic invariance of Maxwell's equations

A simple *Gedankenexperiment* reveals that the constancy of the speed of light indeed requires us to abandon the concept of absolute time. Imagine a "light clock" made up of a photon bouncing back and forth between two parallel mirrors (Fig. VII-4a). The clock "ticks" each time the photon completes a round-trip journey.

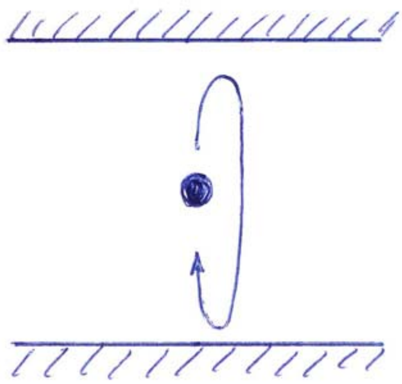

Fig. VII-4a

Let's now imagine that the clock is put on a space shuttle and is moved with respect to us with a high, constant speed. Whilst a passenger of this space shuttle still sees the photon moving perpendicularly to the mirror surfaces in Fig. VII-4a, from our perspective, the photon in the sliding clock must travel at an angle, on a diagonal path (Fig. VII-4b).

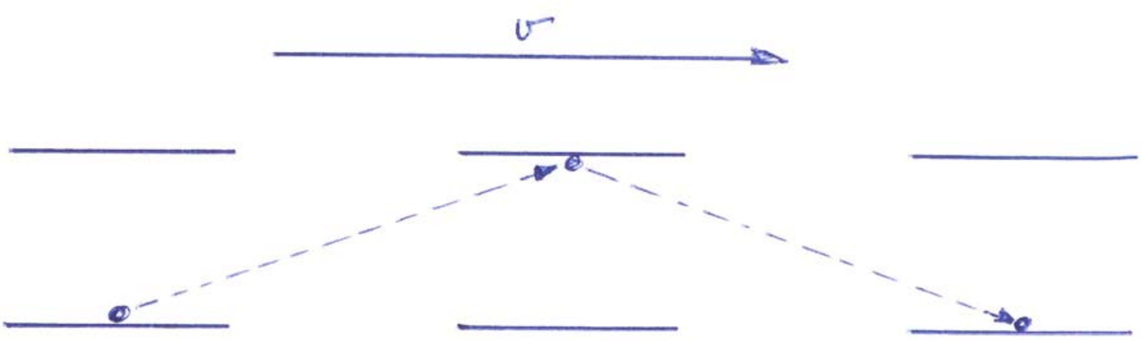

Fig. VII-4b

This path is clearly longer and therefore *because the photon's speed is the same in both frames* we find that, from our perspective the moving clock ticks less frequently. *From our perspective the passage of time is slowed down in a frame moving with respect to us.*

The change in the passage of time is *clearly* inconsistent with Galilean relativity and calls for a modified coordinate transformation. In order to find this new transformation, let us consider two inertial frames of reference K (t, x, y, z) and K' (t', x', y', z'). The coordinate axes in the two frames are parallel and oriented so that K' is moving in the positive z direction with speed v as viewed from K (Fig. VII-5).

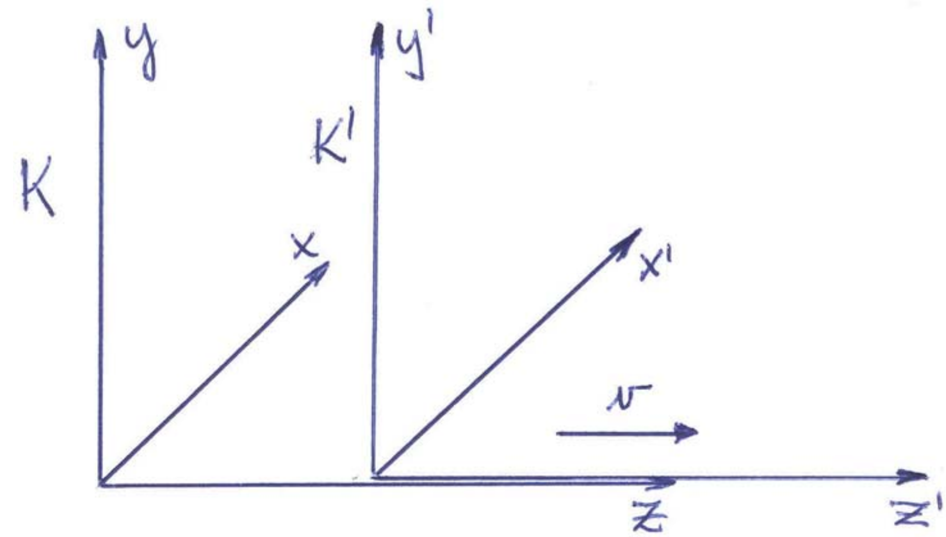

Fig. VII-5

We derive the proper coordinate transformation from the requirement of the invariance of Maxwell's equations under the new transformation. Since there is no relative motion along the *x* and *y* axes, we have *x'* = *x* and *y'* = *y*. Consequently, we can focus on the transformation of the *z* and *t* coordinates.
To this end, we consider a plane electromagnetic wave propagating along the *z* direction, with its electric and magnetic fields polarized along the *x* and *y* direction, respectively:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{e}_x E_x(z,t) = \frac{1}{2}\,\mathbf{e}_x E_0\, e^{i(kz-\omega t)} + c.c. \qquad \text{(VII-15a)}$$

$$\mathbf{B}(\mathbf{r},t) = \mathbf{e}_y B_y(z,t) = \frac{1}{2}\,\mathbf{e}_y B_0\, e^{i(kz-\omega t)} + c.c. \qquad \text{(VII-15b)}$$

The fields must obey the first and second Maxwell's equations (IV-1 and IV-2) in both coordinate systems, hence

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \quad ; \quad \frac{\partial B_y}{\partial z} = -\frac{1}{c^2}\frac{\partial E_x}{\partial t} \qquad \text{(VII-16,17)}$$

$$\frac{\partial E'_x}{\partial z'} = -\frac{\partial B'_y}{\partial t'} \quad ; \quad \frac{\partial B'_y}{\partial z'} = -\frac{1}{c^2}\frac{\partial E'_x}{\partial t'} \qquad \text{(VII-16',17')}$$

The connection between the coordinates *z', t'* and *z, t* must be linear and express that the origin of *K'*, which is defined by *z'* = 0 moves with *v* along the *z* axis of *K.* These conditions are fulfilled by

$$z' = \gamma(z - vt) \qquad \text{(VII-18a)}$$

$$t' = at + bz \qquad \text{(VII-18b)}$$

These relations are indeed linear. It is implicit in (VII-18) that the origins of the spatial coordinates in *K* and *K'* are coincident at $t = t' = 0$. (VII-18a) reveals that $z' = 0$ for $z = vt$, ensuring that the origin of *K'* moves with *v* with respect to *K.* Inverting (VII-18) yields

$$z = \frac{a}{a\gamma + b\gamma v}\,(z' + \frac{\gamma}{a}\,vt') \qquad \text{(VII-19a)}$$

$$t = \frac{\gamma}{a\gamma + b\gamma v}\,t' - \frac{b}{a\gamma + b\gamma v}\,z' \qquad \text{(VII-19b)}$$

We now require – in the spirit of the first postulate – that the form of (VII-19) is identical to that of (VII-18) except for a change in sign in front of *v*, reflecting that from the perspective of an observer residing in *K', K* moves with *v* in the opposite (i.e. negative *z'*) direction. This requirement implies

$$\left.\begin{array}{lcl} \dfrac{a}{a\gamma + b\gamma v} & = & \gamma \\ \dfrac{\gamma}{a} & = & 1 \end{array}\right\} \quad \Rightarrow \quad \left.\begin{array}{l} a = \gamma \\ a\gamma + b\gamma v = 1 \end{array}\right\} \quad \Rightarrow \quad b = -\frac{\gamma^2 - 1}{\gamma v} \qquad \text{(VII-20)}$$

By introducing the notation

$$\alpha = \frac{\gamma^2 - 1}{\gamma^2 v} \qquad \text{(VII-21)}$$

the new coordinate transformation fulfilling the requirements of the first postulate takes the form

$$\begin{array}{l|l} z' = \gamma(z - vt) & z = \gamma(z' + vt') \\ t' = \gamma(t - \alpha z) & t = \gamma(t' + \alpha z') \end{array} \qquad \text{(VII-22,22')}$$

If we now wish to transform Maxwell's equations from *K* to *K'* we will have to make use of

$$\frac{\partial z'}{\partial z} = \gamma \quad ; \quad \frac{\partial z'}{\partial t} = -v\gamma \quad ; \quad \frac{\partial t'}{\partial t} = \gamma \quad ; \quad \frac{\partial t'}{\partial z} = -\alpha\gamma \qquad \text{(VII-23)}$$

which results in

$$\frac{\partial}{\partial z} = \gamma \frac{\partial}{\partial z'} - \alpha\gamma \frac{\partial}{\partial t'}$$
$$\frac{\partial}{\partial t} = \gamma \frac{\partial}{\partial t'} - v\gamma \frac{\partial}{\partial z'} \qquad \text{(VII-24)}$$

Substituting the differential operations (VII-24) into (VII-16) and (VII-17) leads to

$$\frac{\partial}{\partial z'}(\gamma E_x - \gamma v B_y) = -\frac{\partial}{\partial t'}(\gamma B_y - \alpha\gamma E_x) \qquad \text{(VII-16')}$$

$$\frac{\partial}{\partial z'}(\gamma B_y - \frac{\gamma v}{c^2} E_x) = -\frac{1}{c^2}\frac{\partial}{\partial t'}(\gamma E_x - \alpha\gamma c^2 B_y) \qquad \text{(VII-17')}$$

These equations take the same form as the original equations in frame *K* if and only if the expressions in the parentheses can be identified as the transformed field quantities

$$E'_x = \gamma(E_x - vB_y) = \gamma(E_x - \alpha c^2 B_y)$$
$$B'_y = \gamma(B_y - \alpha E_x) = \gamma(B_y - \frac{v}{c^2} E_x) \qquad \text{(VII-25)}$$

which dictates

$$\alpha = \frac{v}{c^2} \qquad \text{(VII-26)}$$

From the requirement of the invariance of Maxwell's equations under the new transformation, we have now obtained the last unknown parameter of the transformation. Substituting (VII-26) into (VII-21) yields

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}}; \qquad \beta = \frac{v}{c} \qquad \text{(VII-27)}$$

With this the *Lorentz transformation* takes the form

$$z' = \gamma(z - vt) \qquad\qquad z = \gamma(z' + vt')$$
$$t' = \gamma(t - \frac{v}{c^2} z) \qquad\qquad t = \gamma(t' + \frac{v}{c^2} z') \tag{VII-28}$$

From our analysis we have also obtained the transformation rules for the electromagnetic fields

$$E'_x = \gamma\,(E_x - vB_y) \qquad ; \qquad B'_y = \gamma\left(B_y - \frac{v}{c^2} E_x \right) \tag{VII-29a}$$

The other transverse components, E_y and B_x , can be obtained by applying the same procedure after changing the polarisation of the plane wave ansatz (VII-15):

$$E'_y = \gamma\,(E_y + v\,B_x) \qquad ; \qquad B'_y = \gamma\,(B_y - \frac{v}{c^2} E_x) \tag{VII-29b}$$

The transformation laws of the *z*-component of the fields will be derived later.