# Quantum states of the electromagnetic field

## Single-mode approximation

In this chapter we shall address some implications of the quantization of the electromagnetic field by confining the discussion to a single mode ( $\mathbf{k}$ , $\sigma$ ) of the radiation field. The selected mode is assumed to propagate along the *z* axis with electric and magnetic fields polarized along the *x* and *y* axes, respectively

$$\hat{E}(z,t) = \sqrt{\frac{2\hbar\omega}{\varepsilon_0 V}} \left(\frac{1}{2} \hat{a} \ e^{i\varphi(z,t)} + \frac{1}{2} \hat{a}^{\dagger} \ e^{-i\varphi(z,t)}\right) \tag{VI-82}$$

$$\hat{B}(z,t) = \sqrt{\frac{2\hbar}{\varepsilon_0 V \omega}} \left( \frac{1}{2} \hat{a} e^{i\varphi(z,t)} + \frac{1}{2} \hat{a}^{\dagger} e^{-i\varphi(z,t)} \right)$$
(VI-83)

with the phase angle

$$\varphi(z,t) = kz - \omega t + \frac{\pi}{2} \tag{VI-84}$$

Here the mode indices have been dropped from the creation and destruction operators for simplicity. They satisfy the commutation rules

$$[\hat{a}, a^{\dagger}] = 1$$
 (VI-85a)

$$[\hat{a}, \hat{H}_{\text{field}}] = \hbar \omega \hat{a} \tag{VI-85b}$$

$$[\hat{a}^{\dagger},\hat{H}_{\text{field}}] = -\hbar\omega\hat{a}^{\dagger} \tag{VI-85c}$$

and their operation on the *photon-number state*  $|n\rangle$  results in

$$\hat{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle \tag{VI-86a}$$

$$\hat{a}^{\dagger} | n \rangle = \sqrt{n+1} | n+1 \rangle \tag{VI-86b}$$

$$\hat{a}^{\dagger}\hat{a}\left|n\right\rangle = n\left|n\right\rangle \tag{VI-86c}$$

From (VI-86b) follows that the number state  $|n\rangle$  can be generated from  $|0\rangle$  by the operator

$$\hat{N}(n)|0\rangle = |n\rangle; \quad \hat{N}(n) = (\hat{a}^{\dagger})^n / \sqrt{n!}$$
(VI-86d)

which is referred to as the number-state creation operator. We can introduce the quadrature operators

$$\hat{X} = \frac{1}{2}(\hat{a} + \hat{a}^{\dagger}) = \sqrt{\frac{\omega}{2\hbar}}\,\hat{q}; \quad \hat{Y} = \frac{i}{2}\,(\hat{a}^{\dagger} - \hat{a}) = \sqrt{\frac{1}{2\hbar\omega}}\hat{p} \tag{VI-87a,b}$$

which obey the commutator relation

$$[\hat{X},\hat{Y}] = \frac{i}{2} \tag{VI-88}$$

and, with the inverse relations

$$\hat{a} = \hat{X} + i\hat{Y}; \quad \hat{a}^{\dagger} = \hat{X} - i\hat{Y} \tag{VI-89a,b}$$

lead to

$$\hat{E}(z,t) = \frac{1}{2}\hat{a}e^{i\phi} + \frac{1}{2}\hat{a}^{\dagger}e^{-i\phi} = \hat{X}\cos\phi - \hat{Y}\sin\phi$$
(VI-90)

if the electric field is measured in units of  $\sqrt{2\hbar\omega/\epsilon_0 V}$ .

### Photon-number states

The photon-number states are the basic states of the quantum theory of light. They form a complete set for the quantum states of a single mode: They are the basis for Planck's black-body distribution but not easy to generate experimentally. Breakthrough experiments were performed by Herbert Walther and co at MPQ over the past decade.

Since

$$\hat{H} = \hbar \omega (\hat{a}^{\dagger} a + \frac{1}{2}) = \hbar \omega (\hat{X}^2 + \hat{Y}^2)$$
(VI-91)

the number state has the quadrature-operator eigenvalue property

$$(\hat{X}^2 + \hat{Y}^2)|n\rangle = (n + \frac{1}{2})|n\rangle$$
 (VI-92)

The quadrature operator expectation values and their variance are (exercise)

$$\left\langle n \right| \hat{X} \left| n \right\rangle = \left\langle n \right| \hat{Y} \left| n \right\rangle = 0 \tag{VI-93}$$

and

$$(\Delta X)^{2} = \left\langle \hat{X}^{2} \right\rangle - \left\langle \hat{X} \right\rangle^{2} = \frac{1}{2} \left( n + \frac{1}{2} \right); \quad (\Delta Y)^{2} = \frac{1}{2} \left( n + \frac{1}{2} \right)$$
(VI-94)



Fig. VI-5

From (VI-90) and (VI-93) follows that the mean field (coherent signal) vanishes in the number state

$$S = \langle E \rangle = \langle n | \hat{E} | n \rangle = 0 \tag{VI-95}$$

and the field variance (noise)

$$N = \left(\Delta E\right)^2 = \left\langle E^2 \right\rangle - \left\langle E \right\rangle^2 = \left\langle n \right| \hat{E}^2 \left| n \right\rangle = \frac{1}{2} \left( n + \frac{1}{2} \right) \tag{VI-96}$$

Fig. VI-6 represents the field properties of the single-mode number state. The vertical axis shows the electric field at some fixed point within a laser resonator versus time.



The field oscillates with an amplitude

$$E_0 = \sqrt{n + \frac{1}{2}}$$
 in units of  $\sqrt{\frac{2\hbar\omega}{\varepsilon_0 V}}$  (VI-97)

which for a typical, meter-scale laser cavity is of the order 10<sup>-4</sup> V/cm. The electric field of the photon number state is represented by a vector of length  $\sqrt{n+1/2}$  and random orientation. As a consequence, the measurement of the electric field may results at any time any value between ~ –  $E_0$  and ~  $E_0$ , with the upper and lower boundary being sharply defined only in the limit of n >> 1. *Phase states* 

Whilst the number state has a fairly well defined amplitude (at least in the limit of n >> 1) we find no vestige of the phase angle  $\varphi$  of the classical wave form. The photon number state  $|n\rangle$  has a completely random phase and, conversely, a state of definite phase is expected to have a completely random photon number distribution. The state associated with phase angle  $\chi_{m}$ , known as the *phase state*, can be constructed as the superposition of photon number states

$$\left|\chi_{m}\right\rangle = \frac{1}{\sqrt{r+1}} \sum_{n=0}^{r} e^{in\chi_{m}} \left|n\right\rangle \tag{VI-98}$$

where

$$\chi_m = 2\pi \frac{m}{r+1}, \quad m = 0, 1, \dots, r$$
 (VI-99)

and *r* is a large number, much larger than the important photon-number constituent in a given field excitation. The phase states form a complete orthonormal set just the photon-number states (**exercise**). A Hermitian operator  $\hat{\chi}$  can be defined<sup>1</sup> such that its expectation values reflect the properties of the classical phase angle and hence this operator can be regarded as the *phase operator*. It can be shown<sup>1</sup> that the states  $|\chi_m\rangle$  introduced in (VI-98) are eigenstates of the quantum mechanical phase operator  $\hat{\chi}$ , i.e. they are states of well-defined phase complementary to the number states  $|n\rangle$ .

### **Coherent states**

In the photon-number state, the classical field strength defined as the expectation value of the electric field operator is zero at any time:

$$E(t) = \langle n | \hat{E}(t) | n \rangle = 0 \tag{VI-100}$$

Experience teaches us that the electric field of an electromagnetic wave often varies sinusoidally with time at a fixed observation point (Fig. VI-7):



Fig. VI-7

Therefore there must be a quantum state of the light field that results in an expectation value of the electric-field operator which reflects this sinusoidal temporal variation. The superposition state

$$\left|\alpha\right\rangle = e^{-\frac{1}{2}\left|\alpha\right|^{2}} \sum_{n=0}^{\infty} \frac{\alpha^{n}}{\sqrt{n!}} \left|n\right\rangle \tag{VI-101}$$

<sup>&</sup>lt;sup>1</sup> D. T. Pegg and S. M. Barnett, Tutorial review: Quantum optical phase, J. Mod. Opt. 44, 225-264 (1997).

can be shown to meet this requirement and is referred to as a *coherent state*. As a matter of fact, as we shall see below, the expectation value of the electric field operator of the coherent state  $|\alpha\rangle$  as

$$\langle E \rangle = \langle \alpha | \hat{E} | \alpha \rangle = \frac{1}{2} \alpha \ e^{i(kz - \omega t + \frac{\pi}{2})} + C.C. = |\alpha| \cos(\varphi + \Theta)$$
(VI-102)

where the angle  $\theta$  is defined by  $\alpha = |\alpha| e^{i\theta}$  and the electric field is measured in units of  $\sqrt{2\hbar\omega/\epsilon_0 V}$ . The properties of coherent states resemble most closely those of a classical electromagnetic wave. Coherent states possess maximum second-order coherence, justifying their name. The single-mode laser operated well above threshold generates such coherent-state excitation of the electromagnetic field.<sup>2</sup> As it is apparent from (VI-102) the complex parameter  $\alpha$  gives the complex amplitude of the classical wave yielded by the expectation value of the electric field in the coherent state. The coherent states form a double continuum parameterised in terms of the real and imaginary parts of  $\alpha$ . They are normalized (exercise)

$$\left\langle \alpha \left| \alpha \right\rangle = e^{-\left| \alpha \right|^2} \sum_{n} \frac{\alpha^{*n} \alpha^n}{n!} = 1$$
(VI-103)

but not orthogonal because (exercise)

$$\left< \alpha \left| \beta \right> = e^{-\frac{1}{2} \left| \alpha \right|^2 - \frac{1}{2} \left| \beta \right|^2} \sum_{n} \frac{\alpha^{*n} \beta^n}{n!} = e^{-\frac{1}{2} \left| \alpha \right|^2 - \frac{1}{2} \left| \beta \right|^2 + \alpha^* \beta}$$
(VI-104)

From (VI-104) follows that

$$\left|\left\langle \alpha \left|\beta\right\rangle\right|^{2} = e^{-\left|\alpha-\beta\right|^{2}} \tag{VI-105}$$

It is apparent from (VI-101) that there is many more coherent states  $|\alpha\rangle$  than number states  $|n\rangle$ , hence the  $|\alpha\rangle$  form an *overcomplete set* of states with their lack of orthogonality being a direct consequence. Nevertheless the states  $|\alpha\rangle$  and  $|\beta\rangle$  become orthogonal in the limit of  $|\alpha - \beta| >> 1$ .

The coherent states are eigenstates of the destruction operator (exercise)

$$\hat{a}|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n} |n-1\rangle = \alpha |\alpha\rangle \qquad (VI-106)$$

consequently the complex number  $\alpha$  is an eigenvalue of the destruction operator. From (VI-106) the conjugate relation follows

$$\langle \alpha | a^{\dagger} = \langle \alpha | \alpha^{\star}$$
 (VI-107)

We can use (VI-86d) to generate the coherent state  $|\alpha\rangle$  from the vacuum state:

<sup>&</sup>lt;sup>2</sup> R. Loudon, *The Quantum Theory of Light,* Oxford University Press, 2000, PP 297-310.

$$\left|\alpha\right\rangle = e^{-\frac{1}{2}\left|\alpha\right|^{2}} \sum_{n} \frac{\left(\alpha \,\hat{a}^{\dagger}\right)^{n}}{n!} \left|0\right\rangle = e^{\alpha \,\hat{a}^{\dagger} - \frac{1}{2}\left|\alpha\right|^{2}} \left|0\right\rangle \tag{VI-108}$$

With the help of the operator identity

$$e^{\hat{b}}e^{\hat{c}} = e^{\hat{b}+\hat{c}+\frac{1}{2}[\hat{b},\hat{c}]}$$
(VI-109)

for any pair of operators  $\hat{b}$  and  $\hat{c}$  that commute with their commutators

$$\left[\hat{b}_{i}\left[\hat{b},\hat{c}\right]\right] = \left[\hat{c}_{i}\left[\hat{b},\hat{c}\right]\right] = 0 \tag{VI-110}$$

and using

$$\hat{a}|0\rangle = 0 \implies e^{-\alpha^*\hat{a}}|0\rangle = |0\rangle$$
 (VI-111)

we obtain (exercise)

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle; \ \hat{D}(\alpha) = e^{\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}}$$
(VI-112)

where  $\hat{D}(\alpha)$  is the *coherent-state displacement operator* creating coherent state  $|\alpha\rangle$  directly from the vacuum state  $|0\rangle$ , analogously to the number-state creation operator  $\hat{N}(n)$  generating number state  $|n\rangle$  from  $|0\rangle$ . Simple inspection yields that

$$D^{\dagger}(\alpha) \hat{D}(\alpha) = \hat{D}(\alpha) D^{\dagger}(\alpha) = 1$$
 (VI-113)

i.e. the displacement operator is a unitary operator.

The expectation value of the number operator in the coherent state  $|\alpha\rangle$  is readily obtained form (VI-106) and (VI-107) as

$$\langle n \rangle = \langle \alpha | \hat{n} | \alpha \rangle = \langle \alpha | \hat{a}^{\dagger} \hat{a} | \alpha \rangle = | \alpha |^{2}$$
 (VI-114)

For calculating the second moment we rearrange  $\hat{a}$  and  $\hat{a}^{\dagger}$  into *normal order* by using their commutator relation (exercise)

$$\hat{n}^2 = \hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}\hat{a} = \hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{a} + \hat{a}^{\dagger}\hat{a}$$
(VI-115)

Normal order means that destruction operators are put to the right of the creation operators. By using the eigenvalue properties (VI-106) and (VI-107) of the coherent states, it is straightforward to evaluate the expectation value of *normally-ordered* operators. This way we obtain

$$\left\langle \hat{n}^{2} \right\rangle = \left\langle \alpha \left| \hat{a}^{\dagger} \hat{a}^{\dagger} \hat{a} \hat{a} + \hat{a}^{\dagger} \hat{a} \right| \alpha \right\rangle = \alpha^{*} \alpha^{*} \alpha \alpha + \alpha^{*} \alpha = \left| \alpha \right|^{4} + \left| \alpha \right|^{2} = \left\langle n \right\rangle^{2} + \left\langle n \right\rangle \quad \text{(VI-116)}$$

yielding the photon-number variance and the fractional uncertainty in the number of photons in the coherent state

$$(\Delta n)^{2} = \left\langle n^{2} \right\rangle - \left\langle n \right\rangle^{2} = \left| \alpha \right|^{2} \tag{VI-117}$$

$$\frac{\Delta n}{\langle n \rangle} = \frac{\left| \alpha \right|}{\left| \alpha \right|^2} = \frac{1}{\left| \alpha \right|}$$
(VI-118)

decreasing with increasing value of the coherent state amplitude  $|\alpha|$ . The probability of finding *n* photons in the coherentstate mode is simply obtained by multiplying  $|\alpha\rangle$  with the bra vector  $\langle n|$  and calculating its modulus square

$$P(n) = \left| \left\langle n \right| \alpha \right\rangle \right|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} = e^{-\left\langle n \right\rangle} \frac{\left\langle n \right\rangle^n}{n!}$$
(VI-119)

This is a Poisson probability distribution with (VI-117) providing the variance of this distribution. For large values of  $\langle n \rangle$  it can be well approximated by a Gaussian distribution

$$P(n) \approx \frac{1}{\sqrt{2\pi \langle n \rangle}} e^{-(n-\langle n \rangle)^2/2\langle n \rangle}$$
(VI-120)

The Poisson photon-number probability distributions of coherent states are shown in Fig. VI-8.



Fig. VI-8

The expectation values of the quadrature operators and the quadrature variances can be readily calculated, again by using the normally-ordered operator formalism

$$\langle \alpha | \hat{X} | \alpha \rangle = \frac{1}{2} \langle \alpha | \hat{a}^{\dagger} + \hat{a} | \alpha \rangle = \frac{1}{2} (\alpha^{*} + \alpha) = |\alpha| \cos \theta$$
 (VI-121a)

- 194 -

and similarly

$$\langle \alpha | \hat{Y} | \alpha \rangle = |\alpha| \sin \theta$$
 (VI-121b)

The same procedure can be applied to the squares of the quadrature operators

$$\hat{X}^{2} = \frac{1}{4} \left( \hat{a}^{\dagger} \hat{a}^{\dagger} + 2\hat{a}^{\dagger} \hat{a} + \hat{a}\hat{a} + 1 \right)$$
(VI-122a)

$$\hat{Y}^{2} = \frac{1}{4} \left( -\hat{a}^{\dagger} \hat{a}^{\dagger} + 2\hat{a}^{\dagger} \hat{a} - \hat{a}\hat{a} + 1 \right)$$
(VI-122b)

resulting in

$$(\Delta X)^2 = (\Delta Y)^2 = \frac{1}{4}$$
(VI-123)

In strong contrast to the respective variances for the number states given by (VI-94), the coherent state is a minimum uncertainty state for all mean photon numbers  $\langle n \rangle = |\alpha|^2$ .

## Classical coherent light field: expectation value of the field operator in the coherent state

With the normally-ordered operator formalism we can also readily calculate the expectation value of the field

$$\langle E \rangle = \langle \alpha | \hat{E} | \alpha \rangle = \frac{1}{2} \langle \alpha | \hat{a}^{\dagger} e^{-i\varphi} + e^{i\varphi} \hat{a} | \alpha \rangle = \frac{1}{2} (\alpha^{*} e^{-i\varphi} + \alpha e^{i\varphi}) = |\alpha| \cos(\varphi + \theta) \quad (\forall I-124)$$

and similarly that of the square of the field

$$\hat{E}^{2} = \frac{1}{4} \left( \hat{a}^{\dagger} \hat{a}^{\dagger} e^{-2i\varphi} + 2\hat{a}^{\dagger} a + \hat{a}\hat{a} e^{2i\varphi} + 1 \right)$$
(VI-125)

## Amplitude noise, phase noise

To obtain the variance of the field

$$N = (\Delta E)^2 = \left\langle E^2 \right\rangle - \left\langle E \right\rangle^2 = \frac{1}{4}$$
(VI-126)

The noise is independent of the mean photon number  $\langle n \rangle = |\alpha|^2$  and has the minimum value that the vacuum state results in

$$\left\langle 0\left|\hat{E}^{2}\right|0\right\rangle - \left\langle 0\left|\hat{E}\right|0\right\rangle^{2} = \frac{1}{4}$$
(VI-127)

independently of the phase angle  $\phi + \theta$ .

For the specific case of a phase angle  $\varphi + \theta = 0$  we have  $\langle E \rangle = |\alpha| = \langle n \rangle^{1/2}$ , from which with the use of (VI-117) we obtain

$$\Delta E \Box \Delta |\alpha| = \Delta \langle n \rangle^{1/2} = \frac{1}{2} \frac{\Delta n}{\langle n \rangle^{1/2}} = \frac{1}{2} \frac{\langle n \rangle^{1/2}}{\langle n \rangle^{1/2}} = \frac{1}{2}$$
(VI-126')

which is in accordance with (VI-126).

Since the uncertainty  $(\Delta E)^2$  is independent of the phase, it can be represented by a circle in Fig. VI-9. The heavy arrow of length  $|\alpha|$  making an angle  $\varphi + \theta$  with the real-field axis shows the coherent state.

The angles  $\phi$  and  $\theta$  have distinctly different physical meaning:  $\phi$  is determined by the position and time of evaluation of field averages and often called *measurement phase angle*, whereas  $\theta$  is a property of the field excitation on which the measurement is performed.



Fig. VI-9

As a consequence of (VI-126), the uncertainty disc has a radius of ½ independently of the amplitude and phase of the coherent state. The orthogonal radii represent the amplitude and phase contributions to the uncertainty in the expectation value of the electric field. The amplitude noise is determined by the uncertainty in the photon number

$$\Delta n = |\alpha|$$

(VI-117')

The phase contribution can be simply estimated on the basis of geometrical arguments apparent from Fig. VI-9, which in the limit of  $|\alpha| >> 1$  yields

$$\Delta \chi = \frac{1}{2|\alpha|} \tag{VI-128}$$

These uncertainties also become apparent in the phase dependence of the expectation value of the electric field of a singlemode coherent state shown in Fig. VI-10.





Eqs. (VI-118) and (VI-128) reveal that both the fractional uncertainty in the photon number and the phase uncertainty scale with  $1/|\alpha|$ . By increasing the mean photon number and thereby the amplitude of the electromagnetic wave, it becomes better defined both in phase and in amplitude. In other words, the corpuscular nature of light becomes less and less pronounced with increasing mean photon number in the mode and hence the description of the field as a classical variable (obeying Maxwell's equations) and the semiclassical theory of light-matter interactions become ever better approximations.

#### Squeezed states

From (VI-117') and (VI-128) the product of the photon-number and phase uncertainties can be obtained as

$$\Delta n \Delta \chi = \frac{1}{2} \tag{VI-129}$$

This result is reminiscent of an uncertainty relationship. Even if it is *not* the consequence of a commutation relation between number and phase operators, it correctly represents the trade-off between the values of the amplitude and phase uncertainties of the electric field of a coherent state. It also suggests that there may be other coherent states of light in which the noise is not evenly distributed among amplitude (photon number) and phase but one is smaller at the expense of the other keeping the value of the uncertainty product in (VI-129) constant. Such states do indeed exist. They are characterized by a field variance that is dependent on the measurement phase angle ,  $\Delta E = \Delta E(\phi)$ , and for some values of  $\phi$  the variance becomes smaller than the minimum value applying to phase-independent variance

$$0 \le (\Delta E(\varphi))^2 < \frac{1}{4}$$

(VI-130)

The field excitation having this property is said to be *quadrature-squeezed*.

Squeezed coherent states<sup>3</sup> are characterized by an elliptic uncertainty disc. Depending on whether the amplitude or phase axis is squeezed, we have to do with amplitude-squeezed (Fig. VI-11) or phase-squeezed (Fig. VI-12) coherent state. The implications of amplitude- and phase-squeezing to the spatial and temporal variation of the field are well revealed by the phase-dependent noise band in Fig. VI-13.



Fig. VI-12

<sup>&</sup>lt;sup>3</sup> For a detailed discussion see e.g. R. Loudon, *The Quantum Theory of Light*, Oxford University Press, 2000.



Fig. VI-13