## VI. Quantum optics

## Quantization of the electromagnetic field

Historically it was found that any attempt of a classical description of the motion electrons in atoms failed. Only quantum mechanics succeeded in describing phenomena as atomic spectra, electron diffraction, electrical conduction in crystals and of central importance for photonics - the interaction of light with matter, including the operation of lasers. In quantum mechanics operators replace the familiar variables of classical theory and the state of the system is replaced by state vectors. This general procedure is not limited to any particular system, but - according to our current insight - must be applied to all classical variables to provide the most correct and most accurate description of nature among all currently available physical theories. According to this general procedure all observable quantities of physics including field variables describing wave phenomena must be accounted for by operators in the same way as we replaced coordinates, momenta or the energy of a mechanical system by operators in the previous chapter.

The quantization of fields, that is the description of field variables by operators leads to quantum field theory, which in case of electromagnetic fields is referred to as quantum electrodynamics. The application of the laws of quantum electrodynamics to optical fields and their interaction with matter has been termed quantum optics. Beyond the consistent extension of the laws of quantum physics from mechanical systems to field variables (last sentence of the previous paragraph) quantum optics does not require any new postulates.

Field quantization is also enforced by experimental evidence. The spectrum of blackbody radiation, the temporal evolution of spontaneous emission as well as the noise characteristics of laser radiation could only be explained in the framework of quantum optics.

## Quantum theory of the harmonic oscillator

The modes of electromagnetic radiation in waveguides as well as in free space can be treated as harmonic oscillators, therefore we address the quantum mechanical description of a harmonic mechanical oscillator before proceeding to field quantization.

The simplest harmonic oscillator is a mass attached to a spring, which provides a restoring force $K x$ proportional to the displacement $x$ of the mass from its equilibrium position (Fig. VI-1).


Fig. VI-1

The Hamiltonian of this mechanical harmonic oscillator is given by

$$
\begin{equation*}
H=\frac{p^{2}}{2 m}+\frac{1}{2} K x^{2} \tag{VI-1}
\end{equation*}
$$

which - with Hamilton's equations of motion (V-1) and (V-2) - leads to the Newton equation for the classical oscillator
$m \frac{d^{2} x}{d t^{2}}=-K x$
with the well-known solution

$$
\begin{equation*}
x=A \sin (\omega t+\varphi) \tag{VI-3}
\end{equation*}
$$

where $A$ is the amplitude, $\varphi$ is the phase and

$$
\begin{equation*}
\omega=\sqrt{\frac{K}{m}} \tag{VI-4}
\end{equation*}
$$

is the (angular) frequency of the oscillation. Expressing $K$ in terms of $m$ and $\omega$, we can rewrite the Hamiltonian of the harmonic oscillator as

$$
\begin{equation*}
H=\frac{1}{2 m}\left(p^{2}+m^{2} \omega^{2} x^{2}\right) \tag{VI-5}
\end{equation*}
$$

The usual way of describing the harmonic oscillator quantum mechanically is to use the Schrödinger representation: replace $p$ by $-i \hbar \partial / \partial x$ in (VI-5) and for obtaining the energy eigenvalues of the oscillator solve

$$
\begin{equation*}
-\frac{\hbar^{2}}{2 m} \frac{\partial^{2} \psi}{\partial x^{2}}+\frac{1}{2} m \omega^{2} x^{2} \psi=E \psi \tag{VI-6}
\end{equation*}
$$

## Second quantization, creation and annihilation operators

The solution of (VI-6) yields not only the energy eigenvalues but also the eigenfunctions $\psi$ in terms of Hermite-Gaussian polynomials. However, we can also treat the harmonic oscillator in a more abstract way without the use of any particular quantum mechanical representation. The following treatment is borrowed from Dirac. ${ }^{1}$ First, we introduce the new operators
$\hat{a}=\frac{1}{\sqrt{2 m \hbar \omega}}(m \omega \hat{x}+i \hat{p})$
$\hat{a}^{\dagger}=\frac{1}{\sqrt{2 m \hbar \omega}}(m \omega \hat{x}-i \hat{p})$

[^0]The operator $\hat{a}^{\dagger}$ is the Hermitian adjoint to $\hat{a}$, since by definition $\hat{p}$ and $\hat{x}$ are Hermitian operators. The coefficients occurring in Eqs. (VI-7) have been chosen to simplify certain relations which follow.

The commutation relation of $\hat{a}$ and $\hat{a}^{\dagger}$

$$
\begin{equation*}
\left[\hat{a}, \hat{a}^{\dagger}\right]=\hat{a} \hat{a}^{\dagger}-\hat{a}^{\dagger} \hat{a}=\frac{i}{\hbar}(\hat{p} \hat{x}-\hat{x} \hat{p})=1 \tag{VI-8}
\end{equation*}
$$

follows (exercise) from that of $\hat{p}$ and $\hat{x}$ given by (V-46) and will be intensively used in the following treatment. Inverting Eqs. (VI-7) yields
$\hat{x}=\sqrt{\frac{\hbar}{2 m \omega}}\left(\hat{a}+\hat{a}^{\dagger}\right)$
$\hat{p}=i \sqrt{\frac{m \hbar \omega}{2}}\left(\hat{a}^{\dagger}-\hat{a}\right)$
and substituting into (VI-5) leads to
$\hat{H}=\frac{1}{2} \hbar \omega\left(a^{\dagger} a+a a^{\dagger}\right)=\hbar \omega\left(a^{\dagger} a+\frac{1}{2}\right)$
and (VI-8) implies that

$$
\begin{equation*}
[\hat{a}, \hat{H}]=\hbar \omega \hat{a} ; \quad\left[\hat{a}^{\dagger}, \hat{H}\right]=-\hbar \omega \hat{a}^{\dagger} \tag{VI-11a,b}
\end{equation*}
$$

The expression of the Hamiltonian in terms of $\hat{a}^{\dagger}$ and $\hat{a}$ is often referred to as second quantization in textbooks. To obtain the energy eigenvalues and eigenvectors of the harmonic oscillator we start out from a particular energy eigenvector $\left|E^{\prime}\right\rangle$ and eigenvalue $E^{\prime}$, apply the commutator of $\hat{a}$ and $\hat{H}$ to this eigenvector and utilize (VI-11a) and $\hat{H}\left|E^{\prime}\right\rangle=E^{\prime}\left|E^{\prime}\right\rangle$ to obtain (exercise)
$\hat{H}\left(\hat{a}\left|E^{\prime}\right\rangle\right)=\left(E^{\prime}-\hbar \omega\right)\left(a\left|E^{\prime}\right\rangle\right)$

If $\left|E^{\prime}\right\rangle$ is an eigenvector of $\hat{H}$ with an eigenvalue $E^{\prime}$ then $\left|E^{\prime \prime}\right\rangle=\hat{a}\left|E^{\prime}\right\rangle$ is also an eigenvector with an eigenvalue $E^{\prime \prime}=E^{\prime}-\hbar \omega$. By repeated operation with $\hat{a}$ on $\left|E^{\prime}\right\rangle$, we find that $\hat{a}^{n}\left|E^{\prime}\right\rangle$ is an eigenvector of $\hat{H}$ with the eigenvalue $E^{\prime}-n \hbar \omega$. For a sufficiently large value of $n$, the eigenvalue appears to become negative. But are negative eigenvalues
possible? To answer this question, let us calculate the expectation value of energy in the energy eigenstate $\left|E^{\prime}\right\rangle$ by using (VI-10)

$$
\begin{equation*}
\left\langle E^{\prime}\right| \hat{H}\left|E^{\prime}\right\rangle=\hbar \omega\left(\left\langle E^{\prime}\right| \hat{a}^{\dagger} \hat{a}\left|E^{\prime}\right\rangle+\frac{1}{2}\left\langle E^{\prime} \mid E^{\prime}\right\rangle\right)=\hbar \omega\left(\left\langle E^{\prime \prime} \mid E^{\prime \prime}\right\rangle+\frac{1}{2}\left\langle E^{\prime} \mid E^{\prime}\right\rangle\right)=E^{\prime} \tag{VI-13}
\end{equation*}
$$

It follows from the definition of the scalar product $(\mathrm{V}-28)$ that the scalar product of a state vector $|\psi\rangle$ with itself cannot be negative. Indeed, by expressing the identity operator (V-42) in terms of the complete orthonormal set $\left|a_{n}\right\rangle$ as $\hat{I}=\sum_{n}\left|a_{n}\right\rangle\left\langle a_{n}\right|$ the scalar product can be reexpressed as

$$
\begin{equation*}
\langle\psi \mid \psi\rangle \equiv\langle\psi| \hat{I}|\psi\rangle=\sum_{n}\left\langle\psi \mid a_{n}\right\rangle\left\langle a_{n} \mid \psi\right\rangle=\sum_{n}\left|\left\langle a_{n} \mid \psi\right\rangle\right|^{2} \geq 0 \tag{VI-14}
\end{equation*}
$$

As a consequence, the eigenvalues of the Hamiltonian of the harmonic oscillator can not be negative.
Hence the reduction of the eigenvalue upon the application of â to $\left|E^{\prime}\right\rangle$ must be terminated. This is only possible if there exists one eigenvector with the property
$\hat{a}\left|E_{0}\right\rangle=0$
Because (VI-15) implies that we can generate no further eigenvectors (with even lower energy) by applying â since $\hat{a}^{n}\left|E_{0}\right\rangle=0$. The eigenvalue of this lowest-energy state, that is the ground state of the harmonic oscillator can be obtained from (VI-13) by taking $\left|E^{\prime}\right\rangle=\left|E_{0}\right\rangle, E^{\prime}=E_{0}$ and, as a consequence of $(\mathrm{VI}-15),\left|E^{\prime \prime}\right\rangle=0$. The result is

$$
\begin{equation*}
E_{0}=\frac{1}{2} \hbar \omega \tag{VI-16}
\end{equation*}
$$

The effect of applying $\hat{a}^{\dagger}$ to $\left|E^{\prime}\right\rangle$ can be derived by using the procedure that leads to (VI-12)
$\hat{H}\left(\hat{a}^{\dagger}\left|E^{\prime}\right\rangle\right)=\left(E^{\prime}+\hbar \omega\right)\left(\hat{a}^{\dagger}\left|E^{\prime}\right\rangle\right)$
that is $\left|E^{\prime \prime \prime}\right\rangle=\hat{a}^{\dagger}\left|E^{\prime}\right\rangle$ is also an eigenvector with an eigenvalue $E^{\prime \prime \prime}=E^{\prime}+\hbar \omega$. By repeated operation with $\hat{a}$ on $\left|E^{\prime}\right\rangle$, we find that $\hat{a}^{n}\left|E^{\prime}\right\rangle$ is an eigenvector of $\hat{H}$ with the eigenvalue $E^{\prime}+n \hbar \omega$. Starting out from the ground state $\left|E_{0}\right\rangle$ we can thus generate new eigenvectors
$\left(a^{\dagger}\right)^{n}\left|E_{0}\right\rangle=\left|E_{n}\right\rangle$
with eigenvalues

$$
\begin{equation*}
E_{n}=\hbar \omega\left(n+\frac{1}{2}\right) \tag{VI-19}
\end{equation*}
$$

Are these all possible eigenvalues and -vectors, or are there others? Repeated application of $\hat{a}$ to any arbitrary $\left|E^{\prime}\right\rangle$ must lead us to $\left|E_{0}\right\rangle$ otherwise we would end up with negative eigenvalues. The eigenvalue of $\left|E_{0}\right\rangle$ must be $1 / 2 \hbar \omega$ according to (VI-13). No matter which eigenvector we start out from, we always end up with the same set of eigenvalues, given by (VI19). Consequently, these eigenvalues must be unique. May there be different vectors $\left|E_{0}\right\rangle$ obeying (VI-15)? If yes, we would have the case of degeneracy since all these eigenvectors must possess the eigenvalue $1 / 2 \hbar \omega$. In the ground state and hence all other energy eigenstates would be degenerate, there would have to be other operators commuting with, but independent of $\hat{H}$ whose eigenvalues would suit for labelling the set of degenerate eigenvectors uniquely (as in the case of the electron moving in the Coulomb potential of the proton in the hydrogen atom the eigenvalues of the angular momentum operator are used to label the degenerate eigenstates of identical energy). In the lack of such an operator in the case of the current problem we conclude that the energy eigenvectors (VI-18) and eigenvalues (VI-19) are unique.


Eq. (VI-19) reveals that the minimum energy of the harmonic oscillator, reached in its ground state $\left|E_{0}\right\rangle$, is $1 / 2 \hbar \omega$, and can only be increased in discrete steps, by the energy quantum $\hbar \omega$ or its integer multiple (Fig. VI-2). The operator $\hat{a}$ is called a destruction or annihilation operator since it destroys one quantum of energy. Its Hermitian conjugate, $\hat{a}^{\dagger}$ is called a creation operator since it creates a quantum of energy.

At the oscillation frequencies of mechanical oscillators $\hbar \omega$ is hardly measurable, hence the mechanical oscillators can - for all practical purposes - be well described by classical physics. However, at optical frequencies $\hbar \omega$ exceeds the work function of specific solids so that the energy quantum, referred to as a photon, becomes easily detectable by utilizing the photoeffect, as we shall see later.

## Fig. VI-2

Are the eigenvectors (VI-18) orthogonal? To answer this question, we calculate the scalar product of to different eigenvectors defined by (VI-18). By making use of the identity

$$
\begin{equation*}
\hat{a}\left(\hat{a}^{\dagger}\right)^{n}=\left(\hat{a}^{\dagger}\right)^{n} \hat{a}+n\left(\hat{a}^{\dagger}\right)^{n-1} \Rightarrow \hat{a}^{m}\left(\hat{a}^{\dagger}\right)^{n}=\hat{a}^{m-1}\left(\hat{a}^{\dagger}\right)^{n} \hat{a}+n \hat{a}^{m-1}\left(\hat{a}^{\dagger}\right)^{n-1} \tag{VI-20a,b}
\end{equation*}
$$

which can be derived by repeated application of the commutation relation (VI-8) (exercise), we find

$$
\begin{equation*}
\left\langle E_{m} \mid E_{n}\right\rangle=\left\langle E_{0}\right| \hat{a}^{m}\left(\hat{a}^{\dagger}\right)^{n}\left|E_{0}\right\rangle=n\left\langle E_{0}\right| \hat{a}^{m-1}\left(\hat{a}^{\dagger}\right)^{n-1}\left|E_{0}\right\rangle \tag{VI-21}
\end{equation*}
$$

By repeated application of this rule we obtain
$\left\langle E_{m} \mid E_{n}\right\rangle=\left\{\begin{array}{cllll}n! & \left\langle E_{0}\right| a^{m-n}\left|E_{0}\right\rangle & =0 & \text { if } & m>n \\ \frac{n!}{(n-m)!} & \left\langle E_{0}\right|\left(a^{\dagger}\right)^{n-m}\left|E_{0}\right\rangle=0 & \text { if } & m<n \\ n! & \left\langle E_{0} \mid E_{0}\right\rangle & =n! & \text { if } & m=n\end{array}\right.$
where we assumed that the ground state $\left|E_{0}\right\rangle$ is normalized, $\left\langle E_{0} \mid E_{0}\right\rangle=1$. From (VI-22) we may conclude that the energy eigenstates (VI-18) are indeed orthogonal as expected for the eigenstates of a Hermitian operator and with the normalization
$\left|E_{n}\right\rangle=\frac{1}{\sqrt{n!}}\left(a^{\dagger}\right)^{n}\left|E_{0}\right\rangle$
form a complete orthonormal set, which we will use throughout the rest of this chapter.

## Number operator, number states

With the help of (VI-20a) we find

$$
\begin{align*}
& \hat{a}\left|E_{n}\right\rangle=\sqrt{n}\left|E_{n-1}\right\rangle  \tag{VI-24}\\
& \hat{a}^{\dagger}\left|E_{n}\right\rangle=\sqrt{n+1}\left|E_{n+1}\right\rangle \tag{VI-25}
\end{align*}
$$

from which it immediately follows that

$$
\begin{equation*}
\hat{a}^{\dagger} \hat{a}\left|E_{n}\right\rangle=n\left|E_{n}\right\rangle \tag{VI-26}
\end{equation*}
$$

that is the operator $\hat{a}^{\dagger} \hat{a}$ counts the number of energy quanta in the energy eigenstates and is therefore referred to as the number operator. Because the energy of the system in state $\left|E_{n}\right\rangle$ consists of $n$ quanta, this state is also referred to as a number state.

The matrix elements of the creation and destruction operators in the $\left|E_{n}\right\rangle$ representation can be written down immediately from (VI-24) and (VI-25) (and utilizing the orthonormality of $\left|E_{n}\right\rangle$ )

$$
\begin{equation*}
a_{m n}=\left\langle E_{m}\right| \hat{a}\left|E_{n}\right\rangle=\sqrt{n} \delta_{m, n-1} \tag{VI-27a}
\end{equation*}
$$

and

$$
\begin{equation*}
a_{m n}^{\dagger}=\left\langle E_{m}\right| \hat{a}^{\dagger}\left|E_{n}\right\rangle=\sqrt{n+1} \delta_{m, n+1} \tag{VI-27b}
\end{equation*}
$$

Furthermore, the matrix representation of the momentum and position operators read as

$$
\begin{equation*}
p_{m n}=\left\langle E_{m}\right| \hat{p}\left|E_{n}\right\rangle=i \sqrt{\frac{m \hbar \omega}{2}}\left(\sqrt{n+1} \delta_{m, n+1}-\sqrt{n} \delta_{m, n-1}\right) \tag{VI-28}
\end{equation*}
$$

and

$$
\begin{equation*}
x_{m n}=\left\langle E_{m}\right| \hat{x}\left|E_{n}\right\rangle=\sqrt{\frac{\hbar}{2 m \omega}}\left(\sqrt{n+1} \delta_{m, n+1}+\sqrt{n} \delta_{m, n-1}\right) \tag{VI-29}
\end{equation*}
$$

## Uncertainty products

The operators of the momentum and position do not commute, hence these quantities can not be measured simultaneously accurately. The uncertainty product $\Delta p \Delta x$ for the energy eigenstates of the harmonic oscillator can be calculated by using (exercise)

$$
\begin{equation*}
(\Delta p)^{2}=\left\langle E_{n}\right|(\hat{p}-\langle p\rangle)^{2}\left|E_{n}\right\rangle=\frac{m \hbar \omega}{2}(2 n+1) \tag{VI-30}
\end{equation*}
$$

and

$$
\begin{equation*}
(\Delta x)^{2}=\left\langle E_{n}\right|(\hat{x}-\langle x\rangle)^{2}\left|E_{n}\right\rangle=\frac{\hbar}{2 m \omega}(2 n+1) \tag{VI-31}
\end{equation*}
$$

where we utilized that - as a consequence of (VI-28) and (VI-29) - the expectation values $\langle p\rangle=\left\langle E_{n}\right| \hat{p}\left|E_{n}\right\rangle=0$ and $\langle x\rangle=\left\langle E_{n}\right| \hat{x}\left|E_{n}\right\rangle=0$. This leads to the uncertainty product

$$
\begin{equation*}
\Delta p \Delta x=\hbar\left(n+\frac{1}{2}\right) \tag{VI-32}
\end{equation*}
$$

According to the Heisenberg uncertainty principle

$$
\begin{equation*}
\Delta p \Delta x \geq \frac{1}{2} \hbar \tag{VI-33}
\end{equation*}
$$

We see that the uncertainty product in the ground state of the harmonic oscillator reaches the smallest possible value allowed by Heisenberg's uncertainty principle. The message (VI-32) conveys is that even in its state of lowest energy the particle does not come to rest, which would imply $\Delta p=0$ and $\Delta x=0$. Instead, it oscillates with a residual energy $1 / 2 \hbar \omega$ around the mean values $\langle p\rangle=0$ and $\langle x\rangle=0$.

The above treatment shows the power of the abstract Dirac formulation of quantum mechanics. Without the use of any particular representation of the Hilbert space of the abstract state vectors, we have been able to derive all results relevant for physical measurements by merely using the respective operators of the physical measurables and their commutation relations. We have now developed the formalism required for quantum optics.

## The quantization of the radiation field in a resonator, definition of the photon

Let us confine a plane optical wave propagating in the $z$ direction between two plane $x-y$ surfaces of perfect conductivity. Such a plane-wave resonator, strictly speaking, would have to be bounded by two plane-parallel mirrors of infinite cross section. This is, of course, not feasible. However, at optical frequencies, the wavelength is many orders of magnitude smaller than a reasonable-sized mirror (say radius $r \approx 1 \mathrm{~cm}$ ). Deviation of the enclosed resonator beam from a plane wave can be quantified by the divergence angle $\theta \approx \lambda / r$, which for visible light $(\lambda \approx 0.5 \mu \mathrm{~m}$ ) amounts to $\theta \approx 50$ microradians (Fig. VI3). Over a propagation length of $L \approx 1 \mathrm{~m}$ this causes an increase of the beam radius by merely $0.5 \%$. Hence the eigenmodes of such a plane-mirror resonator can be well approximated by plane waves of appropriate frequencies (eigenfrequencies of the resonator) thanks to the short wavelength of optical radiation. For quantization purposes, the plane-wave approximation is also applicable to stable, Gaussian-beam resonators, as long as the radial variation of the field is negligible within one wavelength: $\left|\partial F_{00} / \partial r\right| \ll 1 / \lambda$, which implies that the longitudinal field components are negligible in Eqs. (IV-42) and (IV-43).


Fig. VI-3
Since the electric field $\mathbf{E}(\mathbf{r}, t)$ must fulfil $\mathbf{E}(z=0, t)=\mathbf{E}(z=L, t)=0$, the fields inside this plane-wave resonator of volume $V=L A$ with its axis aligned along the $z$ direction can be expanded as

$$
\begin{align*}
& \mathbf{E}(\mathbf{r}, t)=-\sum_{\ell, \sigma} \frac{1}{\sqrt{\varepsilon_{0}}} p_{\ell, \sigma}(t) \mathbf{E}_{\ell, \sigma}(z)  \tag{VI-34}\\
& \mathbf{B}(\mathbf{r}, t)=\sum_{\ell, \sigma} \sqrt{\mu_{0}} \omega_{\ell} q_{\ell, \sigma}(t) \mathbf{B}_{\ell, \sigma}(z)
\end{align*}
$$

where the (standing-wave) field distributions of the $\ell^{\text {th }}$ resonator mode with its electric field polarized along $\mathbf{e}_{\sigma}$ are given by

$$
\begin{equation*}
\mathbf{E}_{\ell, \sigma}(z)=\mathbf{e}_{\sigma} \sqrt{\frac{2}{V}} \sin k_{\ell} z \quad ; \quad \mathbf{B}_{\ell, \sigma}(z)=\mathbf{e}_{\sigma} \times \mathbf{e}_{z} \sqrt{\frac{2}{V}} \cos k_{\ell} z \tag{VI-36}
\end{equation*}
$$

and

$$
\begin{equation*}
k_{\ell}=\frac{2 \pi}{L} \ell ; \quad \sigma=x, y \tag{VI-37}
\end{equation*}
$$

with $\ell$ being a positive integer and $\mathbf{e}_{\sigma}$ denoting the unit vector pointing along the $\sigma$ direction. Here $\mu_{0}=1 / \varepsilon_{0} c^{2}$ is the magnetic permeability of vacuum and $\omega_{\ell}=k_{\ell} / \sqrt{\varepsilon_{0} \mu_{0}}$. The modes are orthogonal, that is - after proper normalization obey

$$
\begin{equation*}
\int_{V} \mathbf{E}_{\ell, \sigma} \cdot \mathbf{E}_{m, \gamma} d^{3} r=\delta_{\ell, m} \delta_{\sigma, \gamma} \tag{VI-38}
\end{equation*}
$$

$$
\int_{V} \mathbf{B}_{\ell, \sigma} \cdot \mathbf{B}_{m, \gamma} d^{3} r=\delta_{\ell, m} \delta_{\sigma, \gamma}
$$

Eqs. (VI-34)-(VI-37) represent the normal mode expansion of the resonator. Substituting (VI-34) and (VI-35) into the first and second Maxwell's equations, (IV-1) and (IV-2), we obtain (exercise)

$$
\begin{align*}
& p_{\ell, \sigma}=\frac{d q_{\ell, \sigma}}{d t}  \tag{VI-39}\\
& \omega_{\ell}^{2} q_{\ell, \sigma}=-\frac{d p_{\ell, \sigma}}{d t} \tag{VI-40}
\end{align*}
$$

which implies

$$
\begin{equation*}
\frac{d^{2} q_{\ell, \sigma}}{d t^{2}}+\omega_{\ell}^{2} q_{\ell, \sigma}=0 \tag{VI-41}
\end{equation*}
$$

Eq. (VI-41) identifies $\omega_{\ell}$ as the oscillation frequency of the $\ell^{\text {th }}$ mode.
The total electromagnetic energy stored in the cavity

$$
\begin{equation*}
H_{\text {field }}=\frac{1}{2} \int_{V}\left(\varepsilon_{0} \mathbf{E}^{2}+\frac{1}{\mu_{0}} \mathbf{B}^{2}\right) d^{3} r \tag{VI-42}
\end{equation*}
$$

which is equivalent to the Hamiltonian of the system, can be expressed in terms of the dynamical variables $p_{\ell, \sigma}(t)$ and $q_{\ell, \sigma}(t)$ by substituting (VI-34) and (VI-35) into (VI-42) and using (VI-38)

$$
\begin{equation*}
H_{\text {field }}=\sum_{\ell, \sigma} \frac{1}{2}\left(p_{\ell, \sigma}^{2}+\omega_{\ell}^{2} q_{\ell, \sigma}^{2}\right) \tag{VI-43}
\end{equation*}
$$

Comparison of (VI-42) with (VI-5) reveals that the electromagnetic field in the resonator behaves mathematically like an ensemble of independent harmonic oscillators. The dynamical variables $p_{\ell, \sigma}(t)$ and $q_{\ell, \sigma}(t)$ constitute the canonically conjugate momentum and position variables, which can be verified by deriving from Hamilton's equations of motion
$\dot{q}_{\ell, \sigma}=\frac{\partial H}{\partial p_{\ell, \sigma}}=p_{\ell, \sigma} ; \quad \dot{p}_{\ell, \sigma}=-\frac{\partial H}{\partial q_{\ell, \sigma}}=-\omega_{\ell}^{2} q_{\ell, \sigma}$

The same equations which we previously obtained from Maxwell's equations for $p_{\ell, \sigma}(t)$ and $q_{\ell, \sigma}(t)$.
We can thus proceed with the quantization precisely in the same manner as we did in the case of the harmonic oscillator, by defining the creation and annihilation operators
$\hat{a}_{\ell, \sigma}^{\dagger}=\frac{1}{\sqrt{2 \hbar \omega_{\ell}}}\left(\omega_{\ell} \hat{q}_{\ell, \sigma}-i \hat{p}_{\ell, \sigma}\right)$
$\hat{a}_{\ell, \sigma}=\frac{1}{\sqrt{2 \hbar \omega_{\ell}}}\left(\omega_{\ell} \hat{q}_{\ell, \sigma}+i \hat{p}_{\ell, \sigma}\right)$
with the commutator relations

$$
\begin{equation*}
\left[\hat{a}_{\ell, \sigma}, \hat{a}_{m, \gamma}\right]=0 ; \quad\left[\hat{a}_{\ell, \sigma}^{\dagger}, \hat{a}_{m, \gamma}^{\dagger}\right]=0 ; \quad\left[\hat{a}_{\ell, \sigma}, \hat{a}_{m, \gamma}^{\dagger}\right]=\delta_{\ell, m} \delta_{\sigma, \gamma} \tag{VI-47}
\end{equation*}
$$

Inverting (V-45) and (VI-46) yields
$\hat{p}_{\ell, \sigma}=i \sqrt{\frac{\hbar \omega_{\ell}}{2}}\left(\hat{a}_{\ell, \sigma}^{\dagger}-\hat{a}_{\ell, \sigma}\right)$
and
$\hat{q}_{\ell, \sigma}=\sqrt{\frac{\hbar}{2 \omega_{\ell}}}\left(\hat{a}_{\ell, \sigma}^{\dagger}+\hat{a}_{\ell, \sigma}\right)$

The operators of the electric and magnetic fields of the resonator in the plane-wave approximation can now be obtained by substituting (VI-48) and (VI-49) into (VI-34),(VI-35)

$$
\begin{align*}
& \hat{\mathbf{E}}=\sum_{\ell, \sigma}-i \mathbf{e}_{\sigma} \sqrt{\frac{\hbar \omega_{\ell}}{\varepsilon_{0} V}}\left(\hat{\mathbf{a}}_{\ell, \sigma}^{\dagger}-\hat{\mathbf{a}}_{\ell, \sigma}\right) \sin k_{\ell} z  \tag{VI-50}\\
& \hat{\mathbf{B}}=\sum_{\ell, \sigma}\left(\mathbf{e}_{\sigma} \times \mathbf{e}_{z}\right) k_{\ell} \sqrt{\frac{\hbar}{\varepsilon_{0} V \omega_{\ell}}}\left(\hat{a}_{\ell, \sigma}^{\dagger}+\hat{a}_{\ell, \sigma}\right) \cos k_{\ell} z \tag{VI-51}
\end{align*}
$$

The summation must be extended to the transverse mode indices if more than one transverse mode is oscillating. The Hamilton operator of the field stored in the cavity can also be expressed in terms of $\hat{a}$ and $\hat{a}^{\dagger}$ by substituting (VI-48) and (VI-49) into (VI-43)

$$
\begin{equation*}
\hat{H}_{\text {field }}=\sum_{\ell, \sigma} \hbar \omega_{\ell}\left(\hat{\lambda}_{\ell, \sigma}^{\dagger} \hat{\ell}_{\ell, \sigma}+\frac{1}{2}\right) \tag{VI-52}
\end{equation*}
$$

If the resonator also contains an atomic system (e.g. gain medium in a laser) interacting with the modes of the resonator, the total Hamiltonian of the system can be written as
$\hat{H}_{\text {total }}=\hat{H}_{\text {field }}+\hat{H}_{\text {electron }}=\hat{H}_{\text {field }}+\hat{H}_{0}+\hat{H}_{\text {int }}$

The state vector of the atom-field system can be expanded in terms of the eigenstates of $\hat{H}_{\text {total }}-\hat{H}_{\text {int }}=\hat{H}_{\text {field }}+\hat{H}_{0}$ :

$$
\begin{equation*}
|\Phi\rangle=\sum_{j, m} c_{j m}\left|\phi_{j m}\right\rangle \quad ; \quad\left|\phi_{j m}\right\rangle=\left|n_{1}, n_{2}, n_{3}, \ldots ., n_{i}, \ldots . .\right\rangle\left|u_{m}\right\rangle \tag{VI-54}
\end{equation*}
$$

where, the index $i$ comprises all mode indices (including the longitudinal mode index $\ell$, the transverse mode indices if apply and the polarization index $\sigma$ ), the index $j$ comprises the photon numbers in all modes and $\left|u_{m}\right\rangle$ is the $m$ th eigenstate of the Hamiltonian of the atom in the absence of fields, $\hat{H}_{0}$. Application of the field and (unperturbed) atomic Hamiltonian to this state vector results in

$$
\begin{equation*}
\hat{H}_{\text {field }}\left|\phi_{j m}\right\rangle=\left\{\sum_{i} \hbar \omega_{i}\left(n_{i}+\frac{1}{2}\right)\right\}\left|\phi_{j m}\right\rangle \quad ; \quad \hat{H}_{0}\left|\phi_{j m}\right\rangle=E_{m}\left|\phi_{j m}\right\rangle \tag{VI-55}
\end{equation*}
$$

indicating that the field energy stored in the $i$ th mode of the resonator is equal to $\hbar \omega_{i}\left(n_{i}+1 / 2\right)$, i.e. to the zero-field energy plus an integer multiple of the elementary excitation of this resonator mode, which has been referred to as a photon. The Hamiltonian (VI-53) together with (V-66), (V-67), and (V-68) provides a full quantum description of light-matter interaction in an optical resonator.

## Travelling-wave quantization

Radiation may interact with matter outside a resonator. In this case, the field can not be expressed in terms harmonic oscillators and the previous approach does not work. Rather, we expand the electromagnetic field in terms of plane waves, which mathematically constitutes a 3-dimensional Fourier expansion (similar to the 2D expansion used in Fourier optics, see. Chapter III). As it is more convenient to handle series rather than integral representations, we introduce the so-called box normalization. In this concept, space is limited to an arbitrary but usually large volume surrounding the region of interest, for example an atomic system and we assume that the field outside is a periodic repetition of the field inside the volume. The use of a square box with sides of equal length

$$
\begin{equation*}
0 \leq x \leq L ; \quad 0 \leq y \leq L ; \quad 0 \leq z \leq L \tag{VI-56}
\end{equation*}
$$

makes it easy to impose this periodic boundary condition. The procedure outlined above is the prescription of a three dimensional Fourier series expansion.

Following this procedure we can expand the vector potential $\mathbf{A}(\mathbf{r}, t)$ into a Fourier series, the components of which are solution of
$\nabla^{2} \mathbf{A}(r, t)-\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}} \mathbf{A}(r, t)=0$
and can be formally regarded as harmonic oscillators of discrete eigenfrequencies, which are defined by the periodic boundary conditions. Eq. (VI-58) is obtained here by using the Coulomb gauge
$\nabla \mathbf{A}(\mathbf{r}, t)=0$
and assuming a time independent scalar potential
$\phi=0$

Under these circumstances (IV-24b) simplifies to (VI-57) in the absence of free currents and we have

$$
\begin{equation*}
\mathbf{B}=\nabla \times \mathbf{A} ; \quad \mathbf{E}=-\frac{\partial}{\partial t} \mathbf{A} \tag{VI-60}
\end{equation*}
$$

implying transverse radiation fields (an assumption also implicit in our procedure for resonator mode quantization). The quantization of general the electromagnetic fields (i.e. those having field components pointing along the wave vector, termed

Iongitudinal field components) should be done in the Lorentz gauge, which is a rather complicated procedure treated in a few books on quantum electrodynamics².
For most problems in quantum optics, it is a good approximation to quantize only the transverse field components obeying (VI-58) and (VI-60) and treat the time-independent scalar potential and its resultant electric field (such as that binding electrons to the nucleus) as a classical, unquantized field incorporated in $\hat{H}_{0}$ of (VI-53). Quantization of the total field makes this force appear as a result of an exchange of virtual "longitudinal" photons between these particles and gives rise to small (but well measurable) effects such as e.g. the Lamb shift, which can not be described by our simplified treatment.

The formal analogy of the terms of the Fourier-series expansion of $\mathbf{A}(\mathbf{r}, t)$ to the description of harmonic oscillators allows again the introduction of the creation and annihilation operators, in terms of which the operator of the vector potential of a transverse radiation field can be expanded as ${ }^{3}$
$\hat{\mathbf{A}}=\sum_{k, \sigma} \mathbf{e}_{\mathbf{k}, \sigma} \sqrt{\frac{\hbar}{2 \varepsilon_{0} V \omega_{k}}}\left(\hat{a}_{\mathbf{k}, \sigma}^{\dagger} e^{-i \mathbf{k r}}+\hat{a}_{\mathbf{k}, \sigma} e^{i \mathbf{k r}}\right)$
where
$\mathbf{k}=\frac{2 \pi}{L}\left(\mathbf{e}_{x} n_{x}+\mathbf{e}_{y} n_{y}+\mathbf{e}_{z} n_{z}\right) ; \quad \omega_{k}=|\mathbf{k}| c ; \quad \sigma=1,2$
with $n_{x}, n_{y}$ and $n_{z}$ being integers. The Hamiltonian takes the usual form

$$
\begin{equation*}
\hat{H}=\sum_{\mathbf{k}, \sigma} \hbar \omega_{k}\left(\hat{a}_{\mathbf{k}, \sigma}^{\dagger} \hat{a}_{\mathbf{k}, \sigma}+\frac{1}{2}\right) \tag{VI-63}
\end{equation*}
$$

The creation and annihilation operators commute according to (VI-45). From Eq. (VI-61) we can derive the electric field and magnetic field operators by using (VI-60):

$$
\begin{align*}
& \hat{\mathbf{E}}=-\frac{\partial}{\partial t} \hat{\mathbf{A}}=-\sum_{\mathbf{k}, \sigma} \mathbf{e}_{\mathbf{k}, \sigma} \sqrt{\frac{\hbar}{2 \varepsilon_{0} V \omega_{k}}}\left(\frac{d \hat{d}_{\mathbf{k}, \sigma}^{\dagger}}{d t} e^{-i \mathbf{k} r}+\frac{d \hat{a}_{\mathbf{k}, \sigma}}{d t} e^{i \mathbf{k} r}\right)  \tag{VI-64}\\
& \hat{\mathbf{B}}=\nabla \times \hat{\mathbf{A}}=-i \sum_{\mathbf{k}, \sigma} \mathbf{k} \times \mathbf{e}_{\mathbf{k}, \sigma} \sqrt{\frac{\hbar}{2 \varepsilon_{0} V \omega_{k}}}\left(\hat{a}_{\mathbf{k}, \sigma}^{\dagger} e^{-i \mathbf{k r}}-\hat{a}_{\mathbf{k}, \sigma} e^{i \mathbf{k r}}\right) \tag{VI-65}
\end{align*}
$$

The temporal derivatives of the creation and annihilation operators can be calculated by applying the operator equation of motion (V-45) and using the commutator relations (VI-11) (exercise)
$\frac{d \hat{a}_{k, \sigma}^{\dagger}}{d t}=\frac{1}{i \hbar}\left[\hat{a}_{\mathrm{k}, \sigma}^{\dagger}, \hat{H}\right]=i \omega_{k} \hat{a}_{\mathrm{k}, \sigma}^{\dagger}$

[^1]$$
\frac{d \hat{a}_{\mathbf{k}, \sigma}}{d t}=\frac{1}{i \hbar}\left[\hat{a}_{\mathbf{k}, \sigma}, \hat{H}\right]=-i \omega_{k} \hat{a}_{\mathbf{k}, \sigma}
$$

The substitution of these expressions into (VI-64) yields

$$
\begin{equation*}
\hat{\mathbf{E}}=-i \sum \mathbf{e}_{\mathbf{k}, \sigma} \sqrt{\frac{\hbar \omega_{k}}{2 \varepsilon_{0} V}}\left(\hat{a}_{\mathbf{k}, \sigma}^{\dagger} e^{-i \mathbf{k r}}-\hat{a}_{\mathbf{k}, \sigma} e^{i \mathbf{k r}}\right) \tag{VI-67}
\end{equation*}
$$

Equations (VI-65) and (VI-67) specify the field operators in the Schrödinger picture, where the state vectors evolve and the operators are "frozen" in time. In quantum optics, it is often more convenient to work in the Heisenberg picture, where the state vectors are independent of time, but the operators evolve according to (VI-66). Eqs. (VI-66) can be readily integrated to yield in the Heisenberg picture
$\hat{a}_{\mathbf{k}, \sigma}(t)=\hat{a}_{\mathbf{k}, \sigma} e^{-i \omega_{k} t}$
$\hat{a}_{\mathbf{k}, \sigma}^{\dagger}(t)=\hat{a}_{\mathbf{k}, \sigma}^{\dagger} e^{i \omega_{k} t}$
Substitution of these time dependent operators into (VI-64) and (VI-65) leads to the field operators in the Heisenberg picture

$$
\begin{align*}
& \hat{\mathbf{E}}(t)=\sum_{\mathbf{k}, \sigma} \frac{1}{2} \mathbf{e}_{\mathbf{k}, \sigma} \sqrt{\frac{2 \hbar \omega_{k}}{\varepsilon_{0} V}}\left(\hat{a}_{\mathbf{k}, \sigma}^{\dagger} e^{-i\left(\mathbf{k r}-\omega_{k} t+\frac{\pi}{2}\right)}+\hat{a}_{\mathbf{k}, \sigma} e^{i\left(\mathbf{k r}-\omega_{k} t+\frac{\pi}{2}\right)}\right)  \tag{VI-69}\\
& \hat{\mathbf{B}}(t)=\sum_{\mathbf{k}, \sigma} \frac{1}{2} \mathbf{k} \times \mathbf{e}_{\mathbf{k}, \sigma} \sqrt{\frac{2 \hbar}{\varepsilon_{0} V \omega_{k}}}\left(\hat{a}_{\mathbf{k}, \sigma}^{\dagger} e^{-i\left(\mathbf{k r}-\omega_{k} t+\frac{\pi}{2}\right)}+\hat{a}_{\mathbf{k}, \sigma} e^{i\left(\mathbf{k r}-\omega_{k} t+\frac{\pi}{2}\right)}\right)
\end{align*}
$$

70) 

## Spontaneous atomic transitions

In the framework of the semiclassical theory of light-matter interactions we have been able to calculate the transition rate of atomic transitions induced by the field. However the rate of spontaneous transitions had to be introduced phenomenologically. With the field quantized, the rate of spontaneous emission can now be readily calculated.

Let us assume that a plane wave propagating along $\mathbf{k}$ with a polarization $\sigma$ interacts with atoms prepared in an excited state $\left|u_{2}\right\rangle$ with energy $E_{2}$ resonantly to induce a transition to a lower state $\left|u_{1}\right\rangle$ with energy $E_{1}$ such that $E_{2}-E_{1}=\hbar \omega_{0} \approx \hbar \omega_{k}$.


Fig. VI-4
As Fermi's golden rule for the transition rate

$$
\begin{equation*}
W_{\text {initial } \rightarrow \text { final }}=\frac{\pi}{2 \hbar}\left|H_{\text {fif }}^{\prime}\right|^{2} \delta\left(E_{\text {final }}-E_{\text {initial }}\right) \tag{VI-71}
\end{equation*}
$$

dictates energy conservation, the mode $(\mathbf{k}, \sigma)$ must gain a quantum upon the atomic transition, i.e. the initial and final states of the field-atom system are, as depicted in Fig. (VI-4):
$\left|\phi_{i}\right\rangle=\left|n_{\mathbf{k}, \sigma}\right\rangle\left|u_{2}\right\rangle ; \quad\left|\phi_{f}\right\rangle=\left|n_{\mathbf{k}, \sigma}+1\right\rangle\left|u_{1}\right\rangle$

The transition rate is driven by the interaction Hamiltonian, which can be written as

$$
\begin{equation*}
\hat{H}_{\text {int }}=\frac{1}{2} \hat{H}^{\prime} e^{-i \omega_{k} t}+\frac{1}{2}\left(\hat{H}^{\prime}\right)^{\dagger} e^{i \omega_{k} t} \tag{VI-73}
\end{equation*}
$$

$H_{f i}^{\prime}$ in Fermi's golden rule stands for the matrix element of $\hat{H}^{\prime}$ or $\left(\hat{H}^{\prime}\right)^{\dagger}$ for and upward or downward transition, respectively (as it is apparent from Eq. V-102 in the derivation of the golden rule). The interaction Hamiltonian describing the interaction between the $(\mathbf{k}, \sigma)$ mode of the field and the atoms

$$
\begin{equation*}
\hat{H}_{\mathrm{int}}=-e \hat{\mathbf{E}}_{\mathbf{k}, \sigma}(\mathbf{r}, t) \cdot \mathbf{r} \tag{VI-74}
\end{equation*}
$$

Substituting (VI-69) into (VI-74) and comparing the latter with (VI-73) yields in the electric dipole approximation [kr $\approx 0$ in (VI-69)]

$$
\begin{equation*}
\left(\hat{H}^{\prime}\right)^{\dagger}=i e \sqrt{\frac{2 \hbar \omega_{k}}{\varepsilon_{0} V}} \hat{a}_{\mathbf{k}, \sigma}^{\dagger} \mathbf{e}_{\mathbf{k}, \sigma} \cdot \mathbf{r} \tag{VI-75}
\end{equation*}
$$

which yields the transition matrix element relevant for downward transition
$H_{\mathrm{fi}}^{\prime}=\left\langle\phi_{f}\right|\left(\hat{H}^{\prime}\right)^{\dagger}\left|\phi_{i}\right\rangle=\left\langle n_{\mathbf{k}, \sigma}+1, u_{1}\right|\left(\hat{H}^{\prime}\right)^{\dagger}\left|n_{\mathbf{k}, \sigma}, u_{2}\right\rangle=$
$=i \sqrt{\frac{2 \hbar \omega_{k}}{\varepsilon_{0} V}} \sqrt{n_{k, \sigma}+1} \mathbf{e}_{\mathbf{k}, \sigma} \cdot \mu$
where $\mu=\left\langle u_{1}\right| e r\left|u_{2}\right\rangle$ is the electric dipole matrix element, see Eq. (V-111), and we utilized $\left\langle n_{\mathbf{k}, \sigma}+1\right| \hat{a}_{\mathbf{k}, \sigma}^{\dagger}\left|n_{\mathbf{k}, \sigma}\right\rangle=\sqrt{n_{\mathbf{k}, \sigma}+1}$, which follows from Eq. (VI-25). With (VI-76) the transition rate induced by one mode of the radiation field is given by

$$
\begin{align*}
& W_{\text {downward/mode }}=\frac{\pi \omega_{k}}{\varepsilon_{0} V}\left(n_{\mathbf{k}, \sigma}+1\right)\left|\mathbf{e}_{\mathbf{k}, \sigma} \cdot \boldsymbol{\mu}\right|^{2} \delta\left(E_{1}-E_{2}+\hbar \omega_{k}\right)=  \tag{VI-77}\\
& =W_{\text {induced/mode }}+W_{\text {spont/mode }}
\end{align*}
$$

where
$W_{\text {induced/mode }}=\frac{\pi \omega_{k}}{\varepsilon_{0}}\left|\mathbf{e}_{\mathrm{k}, \sigma} \cdot \mu\right|^{2} \delta\left(E_{1}-E_{2}+\hbar \omega_{k}\right) \frac{n_{\mathbf{k}, \sigma}}{V}$
$W_{\text {spont/mode }}=\frac{\pi \omega_{k}}{\varepsilon_{0} V}\left|\mathbf{e}_{\mathbf{k}, \sigma} \cdot \mu\right|^{2} \delta\left(E_{1}-E_{2}+\hbar \omega_{k}\right)$
are the transition rate induced by the field and the spontaneous transition rate, respectively. It can be readily shown by the same approach (exercise) that for an upward transition the spontaneous transition rate is zero, that is there is no spontaneous upward transition.

The induced transition rate obtained in (VI-78) can be shown to be equivalent to ( $\mathrm{V}-116$ ) derived within the semiclassical theory by making the replacements $\varepsilon_{0} \rightarrow \varepsilon_{0} \varepsilon_{r}=\varepsilon_{0} n^{2} ; \frac{n_{\mathbf{k}, \sigma}}{V} \rightarrow \frac{F n}{C}$ and $\overline{\left(\mathbf{e}_{\mathrm{k}, \sigma} \cdot \mu\right)^{2}} \rightarrow \frac{1}{3} \mu^{2}$ with the latter implied by ensemble averaging for randomly oriented atoms (exercise).

The spontaneous emission rate can be calculated by summing (VI-79) over all modes of the field ${ }^{4}$

$$
\begin{equation*}
W_{\text {spon }}=\frac{\mu^{2} \omega_{0}^{3} n}{3 \pi c^{3} \varepsilon_{0} \hbar} \tag{VI-80}
\end{equation*}
$$

Summing this for all possible downward transitions from an arbitrary initial eigenstate $\left|u_{k}\right\rangle$ of the atomic Hamiltonian $\hat{H}_{0}$ yields the radiative lifetime of this particular state
$\frac{1}{\tau_{k}^{r}}=\sum_{m=0}^{k-1} W_{\text {spon }, k \rightarrow m}$
which has been introduced phenomenologically in the rate-equation modelling of light-matter interactions.

[^2]
[^0]:    ${ }^{1}$ P. A. M. Dirac, The Principles of Quantum Mechanics, $4^{\text {th }}$ ed. Oxford, 1957.

[^1]:    ${ }^{2}$ A. I. Akhiezer and V. B. Berestetskii, Quantum Electrodynamics (Interscience Publishers, 1965)
    W. Heitler, The Quantum Theory of Radiation 3rd Ed. (Oxford, 1957)
    ${ }^{3}$ D. Marcuse, Principles of Quantum Electronics, Academic Press, Inc., 1980

[^2]:    ${ }^{4}$ The derivation can be found e.g. in A. Yariv, Quantum Electronics, 3d Edition, Wiley \& Sons, 1989, p. 166.

