IV. Electromagnetic optics

Microscopic & macroscopic fields, potentials, waves

Heinrich Hertz' (1857-1894) groundbreaking experiment in 1885 revealed that light is electromagnetic radiation, the theoretical laws of which has been previously introduced by *James Clerk Maxwell* (1831-1879) based on the experiments of *Michael Faraday* (1791-1867). Optical frequencies occupy a band of the electromagnetic spectrum that extends from the infrared through the visible to the ultraviolet (Fig. IV-1).

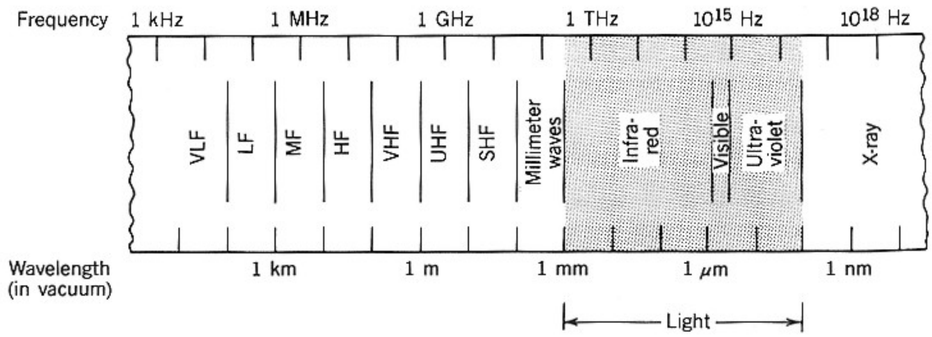


Fig. IV-1

Electromagnetic radiation propagates in the form of two mutually-coupled *vector* waves, an electric-field wave and a magnetic-field wave. The wave optics theory of light addressed in Chapter III is an approximation of the electromagnetic theory describing light phenomena in terms of a single *scalar* function, the wavefunction. This approximation holds for paraxial waves in the absence of polarization effects related to the direction of the electric and magnetic fields. A further simplification leads to ray optics, as discussed before. Thus electromagnetic optics encompasses wave optics, which, in turn, encompasses ray optics. In this chapter we review the basics of electromagnetic theory that are relevant to optics.

Postulates of the electromagnetic theory of light

- Light is electromagnetic radiation described by two related vector fields, the electric and magnetic fields.
- The propagation of light and its emergence due to (microscopic) electric charge and current are described by Maxwell's equations.
- The interaction of light with charged particles is governed by the Lorentz force and preserves energy and momentum.

Maxwell's equations (for time-dependent fields & in the absence of magnetic dipoles)

FARADAY'S LAW
$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) \qquad \text{(IV-1)}$$

AMPÈRE'S LAW (GENERALIZED)
$$\frac{1}{\mu_0}\nabla \times \mathbf{B}(\mathbf{r},t) = \mathbf{J}(\mathbf{r},t) + \varepsilon_0 \frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t) \qquad \text{(IV-2)}$$

COULOMB'S LAW
$$\nabla \mathbf{E}(\mathbf{r},t) = \frac{1}{\varepsilon_0}\rho(\mathbf{r},t) \qquad \text{(IV-3)}$$

ABSENCE OF FREE MAGNETIC POLES $$\nabla \mathrm{B}(\mathrm{r},t) = 0 \tag{IV-4}$$

where the constants ε_0 and $\mu_0 = 1/\varepsilon_0 c^2$ are called the *electric permittivity* and *magnetic permeability* of free space, respectively. Eq. (IV-1) is the differential form of *Faraday's law* of induction, describing the creation of electric field by a time-varying magnetic flux. Eq. (IV-2) is the differential form of the generalized *Ampére's law*, which describes the creation of an induced magnetic flux due to charge flow. Eq. (IV-3) is the differential form of *Coulomb's law*, which describes the relation between the electric field distribution and the charge distribution. Eq. (IV-4) is the mathematical manifestation of the absence of free magnetic monopoles. Eqs. (IV-3) and (IV-4) are also known as *Gauss's law* for the electric and magnetic field, respectively.

Eqs. (IV-1) to (IV-4) are also referred to as the *microscopic* Maxwell's equations, because in this form the charge density $\rho(\mathrm{r},t)$ and the current density $\mathrm{J}(\mathrm{r},t)$ incorporate all *microscopic* contributions

$$\rho(\mathrm{r},t) = \sum_\alpha q_\alpha \delta[\mathrm{r} - \mathrm{r}_\alpha(t)] \tag{IV-5a}$$

$$\mathrm{J}(\mathrm{r},t) = \sum_\alpha q_\alpha \mathrm{v}_\alpha(t)\, \delta[\mathrm{r} - \mathrm{r}_\alpha(t)] \tag{IV-5b}$$

related to the presence and motion of each particle α, having a charge q_α, position $\mathrm{r}_\alpha(t)$, and velocity $\mathrm{v}_\alpha(t)$, irrespective of whether the motion of the particles are induced by the external fields or by other excitations. Hence $\rho(\mathrm{r},t)$ and $\mathrm{J}(\mathrm{r},t)$ may serve either as the source of radiation or be induced by it.

The *Newton-Lorentz equation*

$$F = m_\alpha \frac{d^2}{dt^2}\mathrm{r}_\alpha(t) = q_\alpha \mathrm{E}\,[\mathrm{r}_\alpha(t),t] + q_\alpha \mathrm{v}_\alpha(t) \times \mathrm{B}\,[\mathrm{r}_\alpha(t),t] \tag{IV-6}$$

describes the dynamics of each particle α, having a mass m_α, charge q_α, position $\mathrm{r}_\alpha(t)$, and velocity $\mathrm{v}_\alpha(t)$, under the influence of electric and magnetic forces exerted by the field. Eq. (IV-6) serves for the definition and measurement of the field strengths by means of the electric and magnetic component of the Lorentz force. It also allows to derive the physical units of the field strength from those of the length [meter], mass [kilogram], time [second], and the electric charge [Ampere x second = Coulomb] in the MKSA system of units.

Unit of electric field strength E: Volt/meter [V/m], where $1\ \mathrm{V} = 1\ \mathrm{kg\ m^2\ s^{-3}\ A^{-1}}$

Unit of magnetic-flux density (induction) B: [$\mathrm{Vs/m^2}$]

With these units the *electric permittivity* (also called the *dielectric constant*) and *magnetic permeability* of vacuum, can be expressed as

$\varepsilon_0 \approx 8.85 \times 10^{-12}$ As/Vm $\mu_0 \approx 1.26 \times 10^{-6}$ Vs/Am

Conservation laws, field energy, field momentum, Poynting vector

From (IV-2) and (IV-3) we obtain

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t) = -\nabla \mathbf{J}(\mathbf{r},t) \qquad \text{(IV-7)}$$

the equation of continuity, which expresses the local conservation of electric charge. Hence Maxwell's equations warrant the *conservation of electric charge*. The expression of ρ and J as a function of the particle variables in (IV-5) can be shown to satisfy (IV-7).

A continuity equation analogous to (IV-7) for the energy – after having postulated energy conservation for the interaction of electromagnetic fields with matter (see Postulate #3) – allows to introduce the field energy density and energy flow rate by comparing the new equation with (IV-7). Such a continuity equation for the energy can be derived from Maxwell's equations by using the Lorentz force for describing field-matter interaction. The Lorentz force given by Eq. (IV-6) implies that for particle α, the rate of work done by an external electromagnetic field is $q_\alpha v_\alpha E$. For a current density J Eq. (IV-5b) yields that the rate of work done by the fields per unit volume is JE. By using (IV-2) we can express this as

$$\mathrm{JE} = \varepsilon_0 c^2 \mathrm{E}(\nabla \times \mathrm{B}) - \varepsilon_0 \mathrm{E}\frac{\partial}{\partial t}\mathrm{E} \qquad \text{(IV-8)}$$

By employing the vector identity

$$\nabla(\mathrm{E}\times\mathrm{B}) = \mathbf{B}(\nabla\times\mathrm{E}) - \mathbf{E}(\nabla\times\mathrm{B}) \qquad \text{(IV-9)}$$

and making use of (IV-1) we obtain

$$\mathrm{JE} = -\varepsilon_0 c^2 \nabla(\mathrm{E}\times\mathrm{B}) - \varepsilon_0\left(\mathrm{E}\frac{\partial}{\partial t}\mathrm{E} + c^2\mathrm{B}\frac{\partial}{\partial t}\mathrm{B}\right) \qquad \text{(IV-10)}$$

which can be rewritten as

$$\frac{\partial}{\partial t}\rho_E(\mathbf{r},t) = -\nabla \mathrm{S}(\mathrm{r,t}) - \mathrm{J}(\mathrm{r,t})\,\mathrm{E}(\mathrm{r,t}) \qquad \text{(IV-11)}$$

where

$$\rho_E(\mathbf{r},t) = \frac{1}{2}\varepsilon_0(\mathrm{E}^2 + c^2\mathrm{B}^2) \qquad \text{(IV-12)}$$

and

$$\mathrm{S}(\mathbf{r},t) = \varepsilon_0 c^2 \mathrm{E}\times\mathrm{B} \qquad \text{(IV-13)}$$

If we now require conservation of the total energy of the electromagnetic field + matter, a comparison of (IV-11) with (IV-7) yields that ρ_E must stand for the energy stored in the electromagnetic fields per unit volume (energy density) and the vector S quantifies the direction and amount of field energy flow rate per unit area. It is called the *Poynting vector.*

In a similar way, we can derive a continuity equation for the momentum of the field-matter system. By the same procedure, requiring momentum conservation and comparing the equation with (IV-7) yields for the momentum density of the electromagnetic field:

$$\mathbf{p}(\mathbf{r},t) = \varepsilon_0 \, \mathbf{E} \times \mathbf{B} \qquad \text{(IV-14)}$$

Macroscopic (averaged) fields versus microscopic fields, Maxwell equations for macroscopic fields

In matter, charge is not evenly distributed, rather, it is concentrated in point-like particles. These particles often undergo rapid thermal motion. As a consequence, the microscopic fields produced by these charges vary extremely rapidly in space and time. The spatial variation occur over distances of the order of 10^{-10} m or less, whereas the temporal fluctuations evolve within 10^{-14} – 10^{-13} s (10 - 100 femtoseconds) owing to nuclear vibrations and within 10^{-17} - 10^{-16} s (10 – 100 attoseconds) owing to the motion of electrons. Macroscopic measuring devices usually average over much longer intervals in either space or in time. To predict the result of such a measurement, it is sufficient to average the microscopic fields *spatially* over a volume containing a large number of atoms (this applies well for $L_0^3 \approx 10^{-24}\, m^3$), because over macroscopic distances the microscopic motions are uncorrelated. All that survive are oscillations driven by the external fields. Electromagnetic phenomena can be well described in terms of macroscopic field variables (averaged over atomic length scales) as long as the wavelength of the incident light is longer $L_0 \approx 10 nm$ and field quantities sensed by macroscopic measuring instruments are of interest. X-ray diffraction clearly does not fall into this category. Also, individual molecules in dense matter may feel a field different from the macroscopic field even in the long-wavelength limit, because the polarization of neighbouring molecules gives rise to an internal field E_i in addition to the average macroscopic field E resulting in a total field $\mathrm{E}+\mathrm{E}_i$ at the molecule.[1]

Following the derivation of J. D. Jackson,[2] the spatially-averaged microscopic charge and current density can be expanded into a series of multipoles

$$\left\langle \rho(\mathbf{r},t) \right\rangle = \rho_0(\mathbf{r},t) - \nabla \mathrm{P}(\mathbf{r},t) + \nabla(\nabla \mathrm{Q}) + \ldots, \quad \text{where} \quad \nabla \mathrm{Q} = \sum_j \frac{\partial \mathrm{Q}_{ij}}{\partial r_j} \qquad \text{(IV-a)}$$

$$\left\langle \mathrm{J}(\mathbf{r},t) \right\rangle = \mathrm{J}_0(\mathrm{r},t) + \frac{\partial}{\partial t}\mathrm{P}(\mathrm{r},t) + \frac{1}{\mu_0}\nabla \times \mathrm{M} - \frac{\partial}{\partial t}\nabla \mathrm{Q} + \ldots \qquad \text{(IV-b)}$$

where ρ_0 is the (averaged) *macroscopic charge density*

$$\rho_0(\mathbf{r},t) = \left\langle \sum_{\alpha(\text{free})} q_\alpha \delta(\mathbf{r} - \mathbf{r}_\alpha) + \sum_{n(\text{molecules})} q_n \delta(\mathbf{r} - \mathbf{r}_n) \right\rangle \qquad \text{(IV-c)}$$

[1] J. D. Jackson, Classical Electrodynamics, Third Edition, 1999, John Wiley & Sons, Inc, p. 160.
[2] J. D. Jackson, Classical Electrodynamics, Third Edition, 1999, John Wiley & Sons, Inc, pp. 248-258.

J_0 is the *macroscopic current density*[3]

$$J_0(\mathbf{r},t) = \left\langle \sum_{\alpha(\text{free})} q_\alpha \mathbf{v}_\alpha \delta(\mathbf{r}-\mathbf{r}_\alpha) + \sum_{n(\text{molecules})} q_n \, \mathbf{v}_n \, \delta(\mathbf{r}-\mathbf{r}_n) \right\rangle \tag{IV-d}$$

P is the *macroscopic polarization*

$$\mathbf{P}(\mathbf{r},t) = \left\langle \sum_{n(\text{molecules})} \mathbf{p}_n \, \delta(\mathbf{r}-\mathbf{r}_n) \right\rangle \tag{IV-e}$$

M is the *macroscopic magnetization*

$$\mathbf{M}(\mathbf{r},t) = \left\langle \sum_{n(\text{molecules})} \mathbf{m}_n \, \delta(\mathbf{r}-\mathbf{r}_n) \right\rangle \tag{IV-f}$$

Q is the *macroscopic quadrupole density*

$$Q_{ij}(\mathbf{r},t) = \frac{1}{6}\left\langle \sum_{n(\text{molecules})} (Q_n)_{ij} \delta(\mathbf{r}-\mathbf{r}_n) \right\rangle \tag{IV-g}$$

and the *molecular multipole moments* are given by

MOLECULAR CHARGE $$q_n = \sum_{\alpha(n)} q_\alpha \tag{IV-h}$$

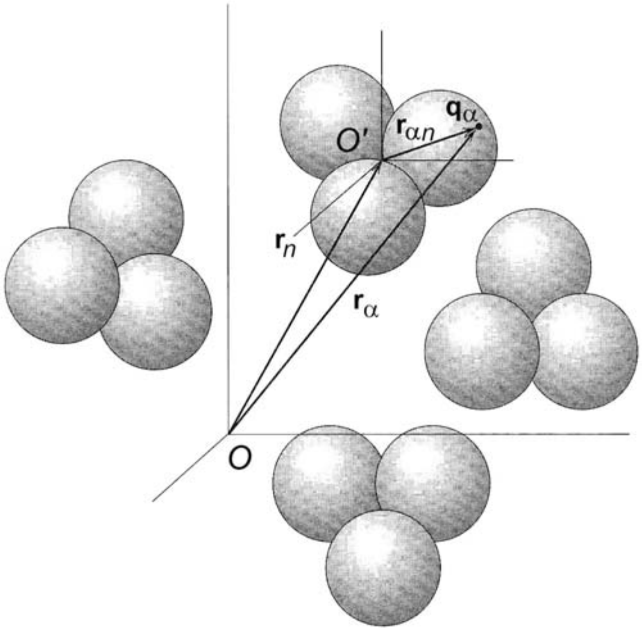
MOLECULAR DIPOLE MOMENT $$\mathbf{p}_n = \sum_{\alpha(n)} q_\alpha \, \mathbf{r}_{\alpha n} \tag{IV-i}$$

MOLECULAR MAGNETIC MOMENT $$\mathbf{m}_n = \sum_{\alpha(n)} \frac{q_\alpha}{2} (\mathbf{r}_{\alpha n} \times \mathbf{v}_{\alpha n}) \tag{IV-j}$$

MOLECULAR QUADRUPOLE MOMENT[4] $$(Q_n)_{ij} = 3\sum_{\alpha(n)} q_\alpha (\mathbf{r}_{\alpha n})_i (\mathbf{r}_{\alpha n})_j \tag{IV-k}$$

---

[3] The subscript “0” will be omitted from ρ_0 and J_0 in later discussions, for simplicity.



Coordinates for the nth molecule. The origin *O'* is fixed in the molecule (usually it is chosen at the centre of mass). The αth charge has coordinate $r_{\alpha n}$ relative to *O*', while the molecule is located relative to the fixed (laboratory) axes by the coordinate r_n.

For the convenient description of electric and magnetic phenomena, the *macroscopic displacement vector*

$$D(\mathbf{r},t) = \varepsilon_0 E(\mathbf{r},t) + P(\mathbf{r},t) - \nabla Q(\mathbf{r},t) + \ldots.. \tag{IV-m}$$

and the *macroscopic magnetic field*

$$H(\mathbf{r},t) = \frac{1}{\mu_0} B(\mathbf{r},t) - \frac{1}{\mu_0} M(\mathbf{r},t) + \ldots \tag{IV-n}$$

have been introduced as auxiliary quantities. The contributions beyond P and M are almost invariably negligible.

Optically-induced charge displacement, generalized polarization, electric dipole approximation, constitutive law

Landau and Lifshitz[5] have pointed out that it is not really meaningful in the optical region to express J and ρ in terms of multipoles as given by (IV-a) and (IV-b), because the usual definition of multipoles are unphysical. It is more expedient to write the macroscopic current as

---

[4] The molecular quadrupole moment has a nonzero trace according to this definition. Making it traceless introduces an additional term in the expression of the macroscopic charge density (see p. 257 in J. D. Jackson, Classical Electrodynamics, Third Edition, 1999)

.
[5] L. D. Landau and E. M. Lifshitz, *Electrodynamics in Continuous Media,* Pergamon Press, New York, 1960, p. 252.

$$\left\langle \mathbf{J}(\mathbf{r},t) \right\rangle = \mathbf{J}_0 + \frac{\partial}{\partial t}\mathbf{P}_{\mathrm{gen}}(\mathbf{r},t) \tag{IV-p}$$

which, with (IV-7), implies

$$\left\langle \rho(\mathbf{r},t) \right\rangle = \rho_0 - \nabla \mathbf{P}_{\mathrm{gen}}(\mathbf{r},\mathrm{t}) \tag{IV-q}$$

where $\mathbf{P}_{gen}$ is referred to as the *generalized electric polarization* incorporating all contributions to a macroscopic displacement of charges driven by the optical fields with $\mathbf{J}_0$ and ρ_0 representing a possible dc current density and a static charge density respectively. Note that $\mathbf{P}_{gen}$ also includes possible contributions of free charges (electrons), the magnitude of the oscillating displacement of which at optical frequencies does not substantially differ from that of their bound counterparts. Eq. (IV-q) ensures that the current induced upon polarizing the medium with the field obeys continuity. With (IV-p,q) and (IV-7) the Maxwell equations take the form

$$\nabla\times\mathbf{E}(\mathbf{r},t) + \frac{\partial}{\partial \mathrm{t}}\mathbf{B}(\mathbf{r},t) = 0 \tag{IV-1'}$$

$$\frac{1}{\mu_0}\nabla\times\mathbf{B}(\mathbf{r},t) - \varepsilon_0\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t) = \mathbf{J}_0 + \frac{\partial}{\partial t}\mathbf{P}_{gen}(\mathbf{r},t) \tag{IV-2'}$$

$$\nabla\left(\varepsilon_0\mathbf{E}(\mathbf{r},t)+\mathbf{P}_{gen}(\mathbf{r},t)\right) = \rho_0 \tag{IV-3'}$$

$$\nabla\,\mathbf{B}(\mathbf{r},t) = 0 \tag{IV-4'}$$

The difference between the *generalized electric polarization* $\mathbf{P}_{gen}$ and the *electric-dipole polarization* P is that $\mathbf{P}_{gen}$ is a *nonlocal* function of the electric field, whereas P is *local*. For optical frequencies and moderate field strengths, the electric-dipole contribution dominates in (IV-a) and (IV-b) and the magnetic dipole and higher-order multipoles can be neglected (Exercise)[6] so that $\mathbf{P}_{gen}$ = P, which we refer to as the **electric dipole approximation**. In what follows, we assume electric dipole approximation, unless otherwise stated.

If a light wave propagates in matter, its electric field tends to induce microscopic dipole moments. The density of these induced atomic or molecular dipoles aligned with the electric field $\mathbf{E}(\mathbf{r},t)$ is referred to as the macroscopic polarization vector $\mathbf{P}(\mathbf{r},t)$. In the case of a linear and instantaneous response (which is a good approximation in the limit of low electric field strengths and far from absorption lines, i.e. resonances, and consequently in the absence of dissipation) the polarization vector is related to the electric field by the linear relationship

$$\mathbf{P}(\mathbf{r},t) = \varepsilon_0\chi\mathbf{E}(\mathbf{r},t) \tag{IV-15}$$

with χ being a second-rank tensor called the *electric susceptibility* tensor (or briefly dielectric tensor). The connection between the polarization and field vector is referred to as the *constitutive law.* Eq. (IV-15) constitutes the constitutive law for a

[6] By using the expressions of the molecular multiple moments given by (IV-i)-(IV-k), show that for $r_{n}/\lambda << 1$ and for $v_{n}/c << 1$, the electric-dipole contribution is dominant in the multipole expansion of the spatially-averaged microscopic charge densities.

medium with *linear, instantaneous response*. For an isotropic medium, the electric susceptibility tensor becomes a scalar quantity χ, implying that the induced dipoles are aligned parallel with the direction of the electric field:

$$\mathrm{P}(\mathbf{r},t) = \varepsilon_0\,\chi\,\mathrm{E}(\mathbf{r},t) \tag{IV-16}$$

The macroscopic current and charge densities in (IV-p) and (IV-q) now takes the form

$$\left\langle \mathrm{J}(\mathbf{r},t)\right\rangle = \mathbf{J}_0 + \frac{\partial}{\partial t}\mathrm{P}(\mathbf{r},t) \tag{IV-17}$$

and

$$\left\langle \rho(\mathbf{r},t)\right\rangle = \rho_0 - \nabla \mathrm{P}(\mathbf{r},t) \tag{IV-18}$$

Replacing now in the continuity equation of energy (IV-10) the microscopic field variables by their macroscopic counterparts as well as the microscopic current density by the macroscopic current density as given by (IV-17) and (IV-16), allows to obtain the continuity equation for the energy in terms of macroscopic field variables (under the assumptions leading to (IV-16):

$$\mathrm{J}_0\mathrm{E} = -\varepsilon_0 c^2 \nabla(\mathrm{E}\times\mathrm{B}) - \left(\varepsilon_0(1+\chi)\mathrm{E}\,\frac{\partial}{\partial t}\,\mathrm{E} + \varepsilon_0 c^2\,\mathrm{B}\frac{\partial}{\partial t}\,\mathrm{B} \right)$$

(IV-10')

which – in the convention that the field does work only on the free electrons – leads to the modified expression for the field energy density

$$\rho_E(\mathbf{r},t) = \frac{1}{2}\,\varepsilon_0(\varepsilon_r \mathrm{E}^2 + c^2\mathrm{B}^2) \tag{IV-12'}$$

where $\varepsilon_r = 1+\chi$ is called the relative permittivity of the medium. Note that in (IV-10') the time average of $J_0E = 0$ if E is a purely optical field without dc component.

Substituting (IV-17) into (IV-2), differentiating the equation with respect to time and expressing the time derivative of B(r,*t*) with E(r,*t*) by using (IV-1) we obtain

$$\nabla \times (\nabla\times \mathbf{E}) + \frac{1}{c^2}\,\frac{\partial^2}{\partial t^2}\mathrm{E} = -\,\mu_0\,\frac{\partial^2}{\partial t^2}\,\mathrm{P} - \mathrm{J}_0 \tag{IV-19}$$

whereas substitution of (IV-18) into (IV-3) yields

$$\nabla(\varepsilon_0\mathbf{E} + \mathrm{P}) = \rho_0 \tag{IV-20}$$

In the absence of dc current and static charge, by use of well-known vector identities[7] Eqs. (IV-19) and (IV-20) result in

$$\nabla^2\mathbf{E}\,(\mathrm{r},t) - \frac{\varepsilon_r(\mathrm{r})}{c^2}\frac{\partial^2}{\partial t^2}\mathrm{E}(\mathrm{r},t) + \nabla\left(\frac{\nabla\varepsilon_r(\mathrm{r})}{\varepsilon_r}\mathrm{E}(\mathrm{r},t)\right) = 0 \qquad \text{(IV-21)}$$

where we have permitted a spatial variation of the electric susceptibility $\chi = \chi(\mathrm{r})$, implying a spatially varying relative permittivity $\varepsilon_r(\mathrm{r}) = 1 + \chi(\mathrm{r})$. If the susceptibility varies in space at a much slower rate than $\mathrm{E}(\mathbf{r},t)$, i.e. $\varepsilon_r(\mathrm{r})$ does not vary significantly within a wavelength distance, the third term in (IV-21) may be neglected in comparison with the first and we obtain the wave equation

$$\nabla^2\mathrm{E}(\mathrm{r},t) - \frac{\varepsilon_r}{c^2}\frac{\partial^2}{\partial t^2}\mathrm{E}(\mathrm{r},t) = 0 \qquad \text{(IV-22)}$$

A similar equation can be derived for the magnetic component of the light fields. A comparison with (III-1) reveals that $n = \varepsilon_r^{1/2}$ is equivalent to the refractive index postulated in ray optics and wave optics. Now, we understand, why we had to postulate n to be a slowly-varying function of r in order for the scalar equation properly describing light wave propagation in the framework of scalar wave theory.

Boundary conditions

In optics, we often encounter situations in which the optical properties (characterized by n) change abruptly across surfaces. From (IV-1) and (IV-4) follows that the tangential components of the electric field E and the normal component of the magnetic field B, respectively, are always continuous functions of position. From (IV-2) in the absence of free currents (including microscopic ones leading to magnetism) and (IV-3) in the absence of free charges it follows that the tangential component of B and the normal component of ε_rE are continuous, respectively.

Vector and scalar potentials, gauge invariance, Lorenz gauge, Coulomb gauge

In regions of space, where the refractive index is continuous, light wave propagation is described by solving (IV-21) or – if $\varepsilon_r(\mathrm{r})$ varies slowly in space – (IV-22). With the electric field wave known, the magnetic field wave can be determined from (IV-1). Here, we introduce, auxiliary field quantities, so-called potentials, which permits an alternative approach and often provide a more convenient means of deriving the electric and magnetic fields of light waves from Maxwell's equations.

Equations (IV-4) and (IV-1) suggest that E and B can be written in the form

$$\mathrm{B}(\mathrm{r},t) = \nabla\times\mathrm{A}(\mathrm{r},\mathrm{t}) \qquad \text{(IV-23a)}$$

---

[7] $\nabla\times(\nabla\times \mathrm{a}) = \nabla(\nabla \mathrm{a}) - \nabla^2\mathrm{a}$
$\nabla a\cdot b = a\nabla b + b\nabla a$

$$\mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) - \nabla\Phi(\mathbf{r},\mathrm{t}) \tag{IV-23b}$$

where A is a vector field, called the *vector potential*, and ϕ is a scalar field, called the *scalar potential.* An obvious benefit from introducing A and ϕ is that Eqs. (IV-1) and (IV-4) are automatically satisfied. Substituting the potentials into (IV-2) and (IV-3) and utilizing once again the same vector identity[7] yields

$$\nabla^2\Phi(\mathbf{r},\mathrm{t}) = -\frac{1}{\varepsilon_0}\rho(\mathbf{r},t) - \nabla\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) \tag{IV-24a}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A}(\mathbf{r},\mathrm{t}) = \mu_0\mathbf{J}(\mathbf{r},t) - \nabla\left[\nabla\mathbf{A}(\mathbf{r},t) + \frac{1}{c^2}\frac{\partial}{\partial t}\phi(\mathbf{r},t)\right] \tag{IV-24b}$$

It follows from (IV-23a) and (IV-23b) that E and B are invariants under the following *gauge transformation*

$$\mathbf{A}(\mathbf{r},t) \rightarrow \mathbf{A}'(\mathbf{r},t) = \mathbf{A}(\mathbf{r},t) + \nabla f(\mathbf{r},\mathrm{t}) \tag{IV-25a}$$

$$\phi(\mathbf{r},t) \rightarrow \phi'(\mathbf{r},t) = \phi(\mathbf{r},t) - \frac{\partial}{\partial t}f(\mathbf{r},t) \tag{IV-25b}$$

where $f(\mathbf{r},t)$ is an arbitrary function of r and *t*. The redundancy in the potentials can be reduced by the choice of the gauge condition which fixes ∇A (the value of ∇x A is already determined by Eq. IV-23a). The most-commonly used gauge is the *Lorenz gauge* defined by

$$\nabla\mathbf{A}(\mathbf{r},t) + \frac{1}{c^2}\frac{\partial}{\partial t}\phi(\mathbf{r},t) = 0 \tag{IV-26}$$

It can be proven that it is always possible to find a function $f(\mathbf{r},t)$ in (IV-25) such that (IV-26) will be satisfied for A' and ϕ'. In the Lorenz gauge, the potential equations take a symmetric form

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi(\mathbf{r},t) = \frac{1}{\varepsilon_0}\rho(\mathbf{r},t) \tag{IV-27a}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A}(\mathbf{r},t) = \mu_0\mathbf{J}(\mathbf{r},t) \tag{IV-27b}$$

which are relativistically invariant, i.e. they keep the same form after a Lorenz transformation. The covariant notation reveals this property automatically. With $\partial_\mu = \{(1/c)(\partial/\partial t), \partial/\partial x, \partial/\partial y, \partial/\partial z\}$ and the four-vectors $A^\mu = \{\Phi/c, A_x, A_y, A_z\}$ and $J^\mu = \{c\rho, J_x, J_y, J_z\}$ associated with the potential and the current, respectively, Eqs. (IV-18) and (IV-19) take the form $\sum \partial_\mu A^\mu = 0$ and $\sum \partial_\nu \partial^\nu A^\mu = (1/\varepsilon_0 c^2) J^\mu$, respectively.

For a medium with refractive index $n = [1 + \chi(\mathrm{r})]^{1/2}$ that varies slowly in space and in the absence of dc current and static charges, by use of (IV-16), (IV-17), (IV-23b) and (IV-24b) we obtain the wave equation for the vector potential

$$\nabla^2 \mathrm{A}(\mathbf{r},t) - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\mathrm{A}(\mathbf{r},t) = 0 \qquad \text{(IV-28)}$$

using the gauge

$$\nabla \mathrm{A} + \frac{n^2}{c^2}\frac{\partial}{\partial t}\phi = 0 \qquad \text{(IV-29)}$$

The standard procedure for describing light wave propagation is now as follows. First we solve the wave equation (IV-28), then we substitute the solution $\mathrm{A}(\mathbf{r},t)$ into (IV-29) to calculate $\phi(\mathbf{r},t)$ and with the potentials known, the electric and magnetic fields can be determined by using (IV-23).

For transverse electromagnetic fields it is often useful to introduce the *Coulomb gauge*

$$\nabla \mathrm{A}(\mathbf{r},\mathrm{t}) = 0 \qquad \text{(IV-30a)}$$

(IV-24a,b) simplify to

$$\nabla^2 \phi = -\frac{1}{\varepsilon_0}\rho(\mathbf{r},t) \qquad \text{(IV-30b)}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathrm{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\phi = \mu_0 \mathbf{J}(\mathbf{r},t) \qquad \text{(IV-30c)}$$

According to Helmholtz's theorem,[8] any vector fields can be written as a sum of two components, one of which has zero divergence and one of which has zero curl. For the current density, the sum is written

$$\mathbf{J} = \mathbf{J}_T + \mathbf{J}_L \qquad \text{(IV-30d)}$$

where

[8] G. B. Arfken, H. J. Weber, *Mathematical Methods for Physicists*, Fourth Edition, Academic Press, San Diego, 1995.

$$\nabla \mathbf{J}_T = 0 \qquad \text{and} \qquad \nabla \times \mathbf{J}_L = 0 \tag{IV-30e}$$

J_T is called the *transverse* or *solenoidal component* and J_L is the *longitudinal* or *irrotational component.* The same definition and nomenclature applies to the other field vectors. For example, it is evident that the magnetic field B is determined by the transverse part of the vector potential, A_T. The Coulomb gauge definition in (IV-30a) identifies A as wholly transverse, with the longitudinal part completely transformed away. With the use of these definitions and the assumption of the Coulomb gauge condition (IV-30a), the complete field equation (IV-30c) is readily separated into its transverse and longitudinal parts as

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla \right) \mathrm{A} = \mu_0 \mathbf{J}_T \tag{IV-30f}$$

and

$$\frac{1}{c^2} \frac{\partial}{\partial t} \nabla \Phi = \mu_0 \, \mathbf{J}_L \tag{IV-30g}$$

The vector potential is thus determined by the transverse part of the current density, whereas the scalar potential satisfies both Eqs. (IV-30b) and (IV-30g), and its elimination from the two gives

$$\nabla \mathbf{J}_L = -\frac{\partial}{\partial t} \rho \tag{IV-30h}$$

which is the equation of charge conservation. The electric field can also be divided into transverse and longitudinal parts:

$$\mathrm{E}_T = -\frac{\partial}{\partial t} \mathrm{A} \qquad \text{and} \qquad \mathrm{E}_L = -\nabla \Phi \tag{IV-30i}$$

whereas the magnetic field B is entirely transverse. The great advantage of the Coulomb gauge for the radiation field and its interaction with charges and currents lies in the clean separation of the field equations into two distinct sets. It is widely used in the quantization of the radiation field.

Plane-wave approximation, energy density and flow rate (intensity) in a light wave, connection between intensity and electric field strength, impedance of free space

Well collimated (paraxial) light beams can often – in first approximation – be treated as plane waves, which can be expressed as

$$\mathrm{A}(\mathbf{r},t)=\frac{1}{2}\,\hat{\mathrm{x}}\left[A_x e^{i(kz-\omega t)}+A_x^* e^{-i(kz-\omega t)}\right] \tag{IV-30}$$

and constitute the simplest solution of the wave equation (IV-28). The dispersion relation $k = \omega n/c$ connects the wavevector **k** (oriented along the *z* axis in the present case) with the oscillation frequency of the wave. Because the divergence of the vector potential A(r,*t*) is zero, (IV-29) yields $\Phi = 0$ and the electric and magnetic field of the plane wave is determined from (IV-23) as

$$\mathrm{E}(\mathbf{r},t)=\frac{1}{2}\,\hat{\mathrm{x}}\left[E_x e^{i(kz-\omega t)}+E_x^* e^{-i(kz-\omega t)}\right] \tag{IV-31a}$$

$$\mathrm{B}(\mathbf{r},t)=\frac{1}{2}\,\hat{\mathrm{y}}\left[B_y e^{i(kz-\omega t)}+B_y^*\, e^{-i(kz-\omega t)}\right] \tag{IV-31b}$$

with the complex field amplitudes given by $E_x = i\omega A_x$ and $B_y = ikA_x$, yielding the connection of

$$B_y = (n/c)\, E_x \tag{IV-31c}$$

between the amplitudes of the magnetic and electric fields. As a consequence, both the electric and the magnetic field of a plane wave is polarized in the plane perpendicular to the direction of propagation, such a wave is referred to as a transverse electromagnetic (TEM) wave.

It is instructive to calculate the ratio of the magnetic to the electric component of the Lorentz force exerted to a charged particle by a plane electromagnetic wave propagating in free space ($n = 1$):

$$F_b/F_e = v_n/c \tag{IV-31d}$$

where v_n is the normal component of the particle's velocity to the magnetic field vector. From (IV-31d) we conclude that the magnetic component of the Lorentz force is negligible compared with the electric component for $v \ll c$ and becomes significant only if the charged particle moves at a speed comparable to that of light. *This implies that the interaction of light with matter can be described in terms of the electric field of a light wave unless the field strength is so high that an electron can acquire a kinetic energy comparable to its rest energy (≈ 0.5 MeV) within one oscillation cycle from the light field.*

Analogously to (IV-30) a plane wave with a vector potential of complex amplitude A_y polarized along the *y* axis also constitutes a plane-wave solution of the wave equation. Because (IV-28) is linear, the (vectorial) sum of the two waves of complex amplitude A_x and A_y also constitutes a solution. The vector potential (just as the electric field vector) of the TEM plane wave

$$\mathrm{A}=\hat{\mathrm{x}}\left|A_x\right|\cos(\varphi_x-\omega t)+\hat{\mathrm{y}}\left|A_y\right|\cos(\varphi_y-\omega t) \tag{IV-32}$$

describes an ellipse as a function of time, at any position in space, the wave is said to be *elliptically polarized.* Important special cases (dictated by the ratio of $|A_x|/|A_y|$ and on the difference $\varphi_x - \varphi_y$) include the *linear* and *circular* polarizations.

The time-averaged value of the field energy density given by (IV-12') of a plane electromagnetic wave can be expressed as

$$\langle \rho_E \rangle = \frac{1}{2}\,\varepsilon_0 \left\langle \varepsilon_r \mathrm{E}^2 + c^2 \mathrm{B}^2 \right\rangle = \frac{1}{4}\varepsilon_0 \varepsilon_r \left|E_x\right|^2 + \frac{1}{4}\varepsilon_0 c^2 \left|B_y\right|^2 =$$
$$= \frac{1}{4}\varepsilon_0 n^2 \left|E_x\right|^2 + \frac{1}{4}\varepsilon_0 c^2 \left|B_y\right|^2 \tag{IV-33a}$$

The time-averaged Poynting vector [see Eq. (IV-13)] gives the direction and magnitude of the time-averaged electromagnetic energy carried by the TEM plane wave through unit cross-sectional area per unit time

$$\langle \mathrm{S} \rangle = \varepsilon_0 c^2 \langle \mathrm{E} \times \mathrm{B} \rangle = \varepsilon_0 c^2 \hat{\mathrm{z}} \left\langle \frac{1}{2}\left(E_x e^{-i\omega t} + E_x^* e^{i\omega t}\right) \frac{1}{2}\left(B_y e^{-i\omega t} + B_y^* e^{i\omega t}\right)\right\rangle =$$
$$= \frac{1}{2}\,\varepsilon_0 c^2 \left|E_x\right|\left|B_y\right| \hat{\mathrm{z}} \tag{IV-33b}$$

By making use of the connection (IV-31c) between the electric and magnetic field amplitude, we find that the time-averaged field energy is evenly distributed between the electric and magnetic fields:

$$\langle \rho_E \rangle = \frac{1}{4}\varepsilon_0 n^2 \left|E_x\right|^2 + \frac{1}{4}\varepsilon_0 c^2 \left|B_y\right|^2 =$$
$$= \frac{1}{4}\varepsilon_0 n^2 \left|E_x\right|^2 + \frac{1}{4}\varepsilon_0 c^2 \frac{n^2}{c^2}\left|E_x\right|^2 = \frac{1}{2}\varepsilon_0 n^2 \left|E_x\right|^2 \tag{IV-34a}$$

and the TEM plane wave carries energy along the direction of its wave vector **k** with a time-averaged intensity

$$I = \frac{1}{2}\,\varepsilon_0 c^2 \left|E_x\right| \left|B_y\right| = \frac{1}{2}\,\varepsilon_0 n c \left|E_x\right|^2 = \frac{c}{n}\langle \rho_E \rangle = \frac{1}{2}\,\frac{\left|E_x\right|^2}{Z} \tag{IV-34b}$$

where

$$Z = \frac{Z_0}{n} \tag{IV-35}$$

is known as the impedance of the medium and

$$Z_0 = \frac{1}{\varepsilon_0 c} \approx 120\pi \frac{V}{A} \approx 377\Omega \tag{IV-36}$$

is the impedance of free space. This simple formula is analogous to the expression of the power *P* dissipated by a sinusoidal voltage of amplitude U_0 applied to a resistance *R*

$$P = \frac{1}{2}\frac{U_0^2}{R} \tag{IV-37}$$

and with the value of the electric field amplitude $|E_x|$ substituted in units of V/cm yields the intensity in units of W/cm^2.

Paraxial electromagnetic waves, relation between electromagnetic optics and scalar wave optics

The paraxial wave is a wave whose wavefront normals make small angles with the optical axis. There exist electromagnetic waves with this property, too. The vector potential of such a wave can be modelled as

$$\mathrm{A}(\mathrm{r},t) = \hat{\mathrm{n}}u(\mathrm{r},t) = \frac{1}{2}\hat{\mathrm{n}}\,F(\mathrm{r})\,e^{i(kz-\omega t)} + c.c. \tag{IV-38}$$

where n is a unit vector in some direction in the xy plane and F(r) fulfils the paraxial approximation (III-19) and obeys the paraxial scalar wave equation (III-20). If so, the paraxial vector wave given by (IV-38) will obey the paraxial vector wave equation that can be derived from (IV-28) in the same way as the paraxial scalar wave equation (III-20) was derived from the scalar wave equation.
With the solution (IV-38) we can construct the electric field of a paraxial light wave by first making use of (IV-29) to yield

$$\phi = \frac{c^2}{i\omega n^2}\nabla \mathrm{A} \tag{IV-39}$$

and then substitute (IV-39) into (IV-23b)

$$\mathrm{E}(\mathrm{r},t) = i\omega\left[\mathrm{A} + \frac{1}{k^2}\nabla(\nabla \mathrm{A})\right] + c.c. \tag{IV-40}$$

where $k = \omega n/c = 2\pi/\lambda$. If A is transverse to the z-direction, the electric field vector is polarized mainly in the x-y plane with a small longitudinal component arising from the second term in the brackets of (IV-40).

The paraxial vector wave given by (IV-38) behaves locally as a TEM plane wave carries energy approximately parallel to the optical axis, the intensity $I \approx |E_x|^2/2Z$. A scalar wave of complex amplitude $U = |E_x|/(2Z)^{1/2}$ may be associated with the paraxial electromagnetic wave so that the two waves have the same intensity and the same wavefronts. The scalar description is an adequate approximation for treating problems of interference, diffraction and propagation of paraxial waves when polarization is not a factor. Take, for example, the Gaussian beam with small divergence angle. Most questions regarding the intensity, focusing by lenses, mirrors, interference may be satisfactorily addressed within the frame of scalar wave theory. However, U and E do not satisfy the same boundary conditions. Problems involving reflection or refraction at boundaries, transmission of light through dielectric waveguides, or questions about the direction of fields naturally call for the electromagnetic theory of light. But electromagnetic theory offers more than just extend the number of light phenomena that can be adequately described. It also simplifies the theory of light in that the number of new postulates (beyond those being used in other fields) has been reduced. For instance, electromagnetic theory of light sheds light on the origin of the refractive index (the polarizability of matter). This opens the way to developing models for calculating the refractive index for any material of known composition on the basis of a unified microscopic model of matter. This unified microscopic model will be the quantum theory of the electron, to be discussed later. As a further simplification, the intensity of a light wave does not have to be postulated (as it was done in scalar wave theory) but follows from requiring energy conservation.

Electric and magnetic fields of a Gaussian beam

The paraxial vector wave given by (IV-38) describes a Gaussian electromagnetic beam if $F(\mathrm{r}) = F_{\mathrm{Gaussian}}(\mathrm{r})$ from (III-44). Supposing that the electric field of the Gaussian beam is (dominantly) polarized along the along the *x*-axis, the vector potential of the Gaussian beam can be written as

$$A_{\mathrm{Gaussian}}(\mathrm{r},t) \equiv A_{00}(\mathrm{r},t) = \frac{1}{2}\hat{\mathrm{x}}\, F_{00}(\mathrm{r})\, e^{i(kz-\omega t)} + c.c. \qquad \text{(IV-41)}$$

where $F_{00}(\mathrm{r}) = F_{\mathrm{Gaussian}}(\mathrm{r})$, as given by Eqs. (III-44)-(III-48), with the subscripts referring to the lowest-order Hermite-Gaussian beam. The magnetic field of the beam can be calculated (exercise) by using (IV-23a):

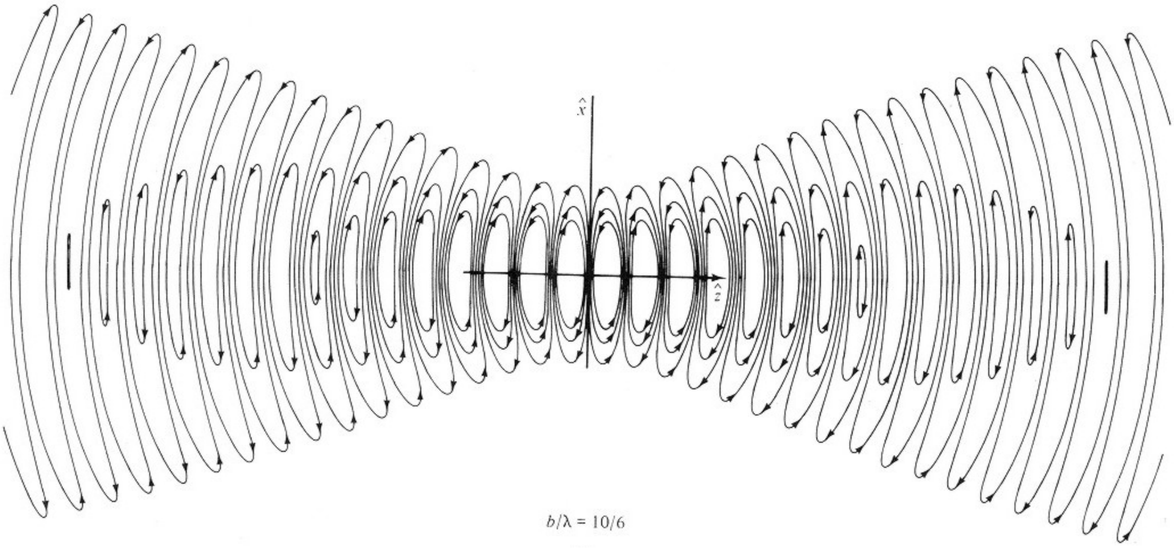
$$\mathrm{B}_{00}(\mathrm{r},t) = \frac{1}{2}\nabla \times \left[\hat{\mathrm{x}} F_{00}(\mathrm{r})\, e^{ikz}\right] e^{-i\omega t} + c.c.$$

$$= \frac{1}{2}\, ik \left[\hat{\mathrm{y}} F_{00}(\mathrm{r}) + \hat{\mathrm{z}} \frac{1}{ik}\frac{\partial F_{00}(\mathrm{r})}{\partial y}\right] e^{i(kz-\omega t)} + c.c. \qquad \text{(IV-42)}$$

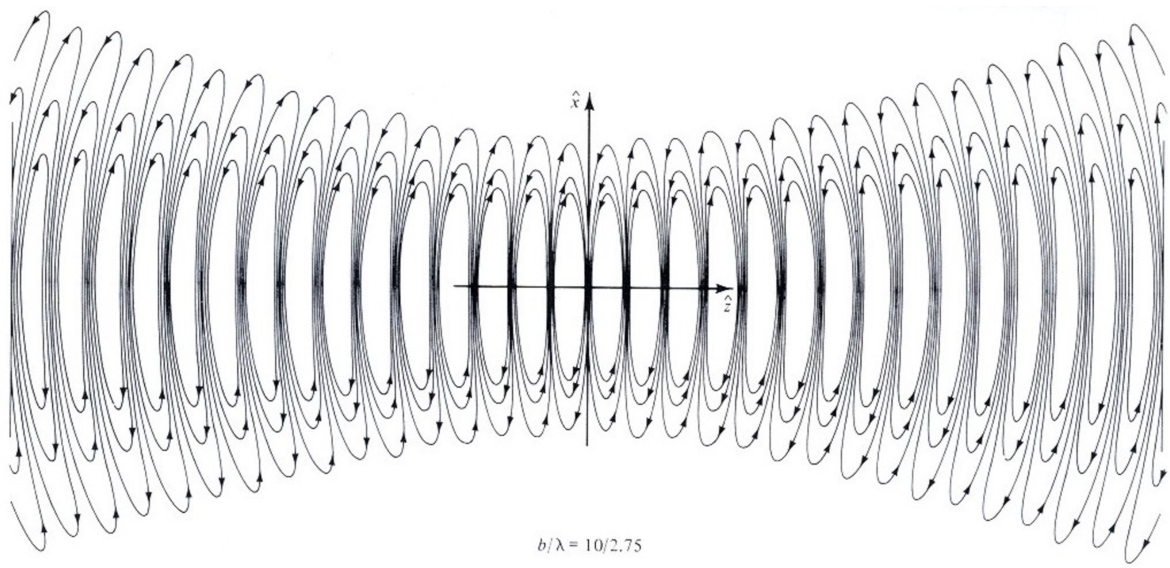
where we have ignored $\partial F_{00}(\mathrm{r})/\partial z$ compared with $kF_{00}(\mathrm{r})$ in the spirit of the slowly-varying envelope (or paraxial) approximation. The magnetic field is polarized primarily along the *y* axis but has a small *z* component as dictated by Gauss's law (IV-4). By use of (IV-40) we obtain (exercise) the electric field – to the same degree of approximation – as

$$\mathrm{E}_{00}(\mathrm{r},t) = \frac{1}{2} i\omega \left[\hat{\mathrm{x}} F_{00}(\mathrm{r}) + \hat{\mathrm{z}} \frac{1}{ik}\frac{\partial F_{00}(\mathrm{r})}{\partial x}\right] e^{i(kz-\omega t)} + c.c. \qquad \text{(IV-43)}$$

Expressions (IV-42) and (IV-43) also apply to higher-order Hermite-Gaussian beams by simply replacing $F_{00}(\mathrm{r})$ with $F_{lm}(\mathrm{r})$ for the beam of order *l,m* in the above expressions. Two plots of E at constant *t* are shown in Fig. IV-2 for the Gaussian beam for two different values of the normalized beam radius w_0/λ at the beam waist.



$2w_0 \approx \lambda$



$2w_0 \approx 1.5\lambda$

Fig. IV-2

Microscopic radiation sources, the field of an oscillating dipole

So far we have been concerned with the solution of Maxwell's equations in the absence of free charge and free currents. These constitute the sources of the fields described by the scalar and vector potentials in Eqs. (IV-27). Their temporal variation is responsible for electromagnetic radiation. In what follows, we shall focus on the implications of temporally-varying charge and current densities localized to a small volume in space. Outside this volume the scalar potential Φ can only be static because of the conservation of charge within the volume and hence radiation can only emerge through the vector potential being induced by the temporally varying current density in (IV-27b). Using Green's theorem it can be shown[9] that in the absence of boundaries, the solution of the inhomogeneous differential equation (IV-27b) takes the form

$$\mathrm{A}(\mathrm{r},t) = \frac{1}{4\pi\varepsilon_0 c^2} \int d^3r' \int dt' \, \frac{\mathrm{J}(\mathrm{r}',t')}{|\mathrm{r}-\mathrm{r}'|} \, \delta\left(t' + \frac{|\mathrm{r}-\mathrm{r}'|}{c} - t\right) \tag{IV-45}$$

The Dirac δ-function ensures causality. For a harmonically-varying current density

$$\mathrm{J}(\mathrm{r},t) = \frac{1}{2}\mathrm{J}(\mathrm{r})e^{-i\omega t} + c.c. \tag{IV-46}$$

the vector potential takes the form

$$\mathrm{A}(\mathrm{r},t) = \frac{1}{2}\mathrm{A}(\mathrm{r})e^{-i\omega t} + c.c. \tag{IV-47}$$

with its complex amplitude given by

$$\mathrm{A}(\mathrm{r}) = \frac{1}{4\pi\varepsilon_0 c^2} \int \mathrm{J}(\mathrm{r}') \, \frac{e^{ik|\mathrm{r}-\mathrm{r}'|}}{|\mathrm{r}-\mathrm{r}'|} d^3r' \tag{IV-48}$$

where $k = \omega/c$. Having determined A(r), the complex amplitudes of the emerging harmonically-oscillating magnetic and electric fields can be obtained from (IV-23a) and (IV-2) outside the radiating volume (where J = 0) as

$$\mathrm{B}(\mathrm{r}) = \nabla \times \mathrm{A}; \qquad \mathrm{E}(\mathrm{r}) = \frac{ic^2}{\omega}\nabla \times \mathrm{B}(\mathrm{r}) \tag{IV-49a,b}$$

In what follows we focus on the case of a source smaller than the wavelength: $kd < 1$, where d stands for the characteristic linear size of the source. At a distance $|\mathrm{r}| = r$ (from the source) large compared with the wavelength, $kr >> 1$, which is called the *far field*, we can utilize in the exponent of (IV-48) the approximation

$$|\mathrm{r}-\mathrm{r}'| \approx r - \mathrm{n}\mathrm{r}' \tag{IV-50}$$

[9] J. D. Jackson, Classical Electrodynamics, Second Edition, 1975, John Wiley & Sons, Inc.

(where n = r/*r* is the unit vector aligned with r), whereas in the denominator we may use

$$\left|\mathbf{r}-\mathbf{r}'\right| \approx r \tag{IV-51}$$

assuming that the source is at the origin of the coordinate system, i.e. $r' < d$. The use of these approximations in (IV-48) lead to

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0 c^2}\,\frac{e^{ikr}}{r}\int \mathbf{J}(\mathbf{r}')\,e^{-ik\mathbf{n}\mathbf{r}'}d^3r' \tag{IV-52}$$

If the size of the source is small compared with the wavelength, the exponent can be expanded in a Taylor series, yielding

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0 c^2}\,\frac{e^{ikr}}{r}\sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}\int \mathbf{J}(\mathbf{r}')(\mathbf{n}\mathbf{r}')^n\,d^3r' \tag{IV-53}$$

Equation (IV-53) is referred to as the *multipole expansion* of localized current distribution. If $kd \ll 1$ applies, only the first few terms (the lowest-order multipoles) in the expansion make significant contribution to the radiated fields, because the n^{th} term in the series scales with $(kd)^n$. The lowest-order term in (IV-53) is given by

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0 c^2}\,\frac{e^{ikr}}{r}\int \mathbf{J}(\mathbf{r}')\,d^3r' \tag{IV-54}$$

and is valid not only in the far field, but anywhere outside the source.
A simple partial integration yields

$$\int \mathbf{J}(\mathbf{r}')\,d^3r' = -\int \mathbf{r}'\,\left[\nabla'\mathbf{J}(\mathbf{r}')\right]d^3r' = -i\omega\int \mathbf{r}'\rho(\mathbf{r}')d^3r' \tag{IV-55}$$

where we have introduced the complex amplitude ρ(r) of the oscillating charge ρ(r,*t*) = ½ ρ(r)exp(-*iωt*) + c.c. and utilized that it obeys the continuity equation:

$$i\omega\rho(\mathbf{r}) = \nabla\mathbf{J}(\mathbf{r}) \tag{IV-56}$$

The complex amplitude of the vector potential now takes the form

$$\mathbf{A}(\mathbf{r}) = \frac{-i\omega}{4\pi\varepsilon_0 c^2}\,\mathbf{p}\,\frac{e^{ikr}}{r} \tag{IV-57}$$

where

$$\mathbf{p} = \int \mathbf{r}'\rho(\mathbf{r}')\,d^3r' \tag{IV-58}$$

is the complex amplitude of the *electric dipole moment* vector of the oscillating charge distribution. By using (IV-49) the electric and magnetic fields of the oscillating electric dipole can be expressed as

$$\mathbf{B}(\mathbf{r}) = \frac{\omega^2}{4\pi\varepsilon_0 c^3} (\mathbf{n}\times\mathbf{p}) \left(1-\frac{1}{ikr}\right) \frac{e^{ikr}}{r} \tag{IV-59a}$$

$$\mathbf{E}(\mathbf{r}) = \frac{\omega^2}{4\pi\varepsilon_0 c^2} (\mathbf{n}\times\mathbf{p})\times\mathbf{n} \; \frac{e^{ikr}}{r} + \frac{1}{4\pi\varepsilon_0}[3\mathbf{n}(\mathbf{n}\mathbf{p})-\mathbf{p}] \left(\frac{1}{r^3}-\frac{ik}{r^2}\right) e^{ikr} \tag{IV-59b}$$

The magnetic field is always normal to the **r** vector, whereas the electric field also has a component parallel to it. In the far field

$$\mathbf{B}(\mathbf{r}) = \frac{\omega^2}{4\pi\varepsilon_0 c^3} (\mathbf{n}\times\mathbf{p}) \; \frac{e^{ikr}}{r} \tag{IV-60a}$$

$$\mathbf{E}(\mathbf{r}) = c\,\mathbf{B}(\mathbf{r})\times\mathbf{n} \tag{IV-60b}$$

both fields are orthogonal to the radial direction as shown in Fig. IV-3.

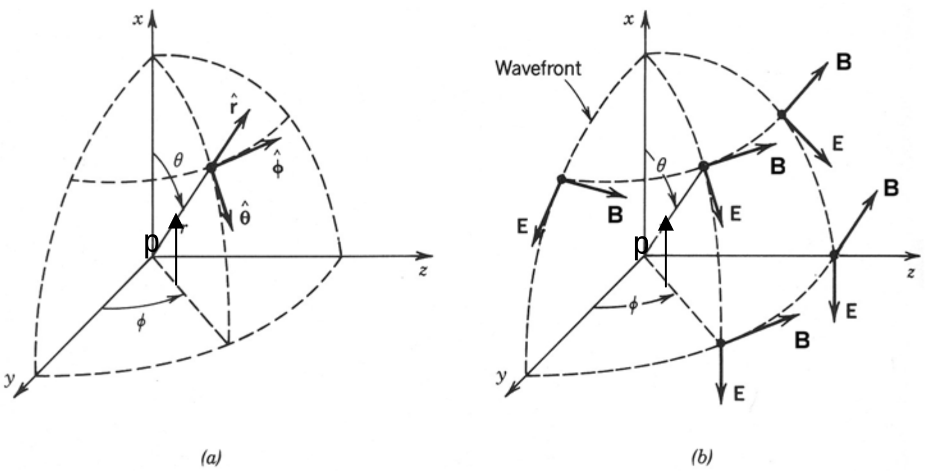


Fig. IV-3

Fig. IV-4 shows the electric field lines of the solution (IV-59b) for the electric dipole moment positioned at the origin of the Cartesian coordinate system and aligned parallel with the *x* axis. The plots (a) and (b) show the field at two time instants, separated by $\Delta t = \pi/2\omega$. The resemblance to the Gaussian beam solution is unmistakable. This is not a surprise, because the Gaussian beam solution is obtained from the radiating dipole solution by applying the paraxial approximation and by removing the singularity at the origin by an imaginary translation of the source.

The next higher-order term in the expansion (IV-53) for $n = 1$ can be shown to result in contribution from the magnetic dipole moment and the electric quadrupole moment of the oscillating charge distribution.

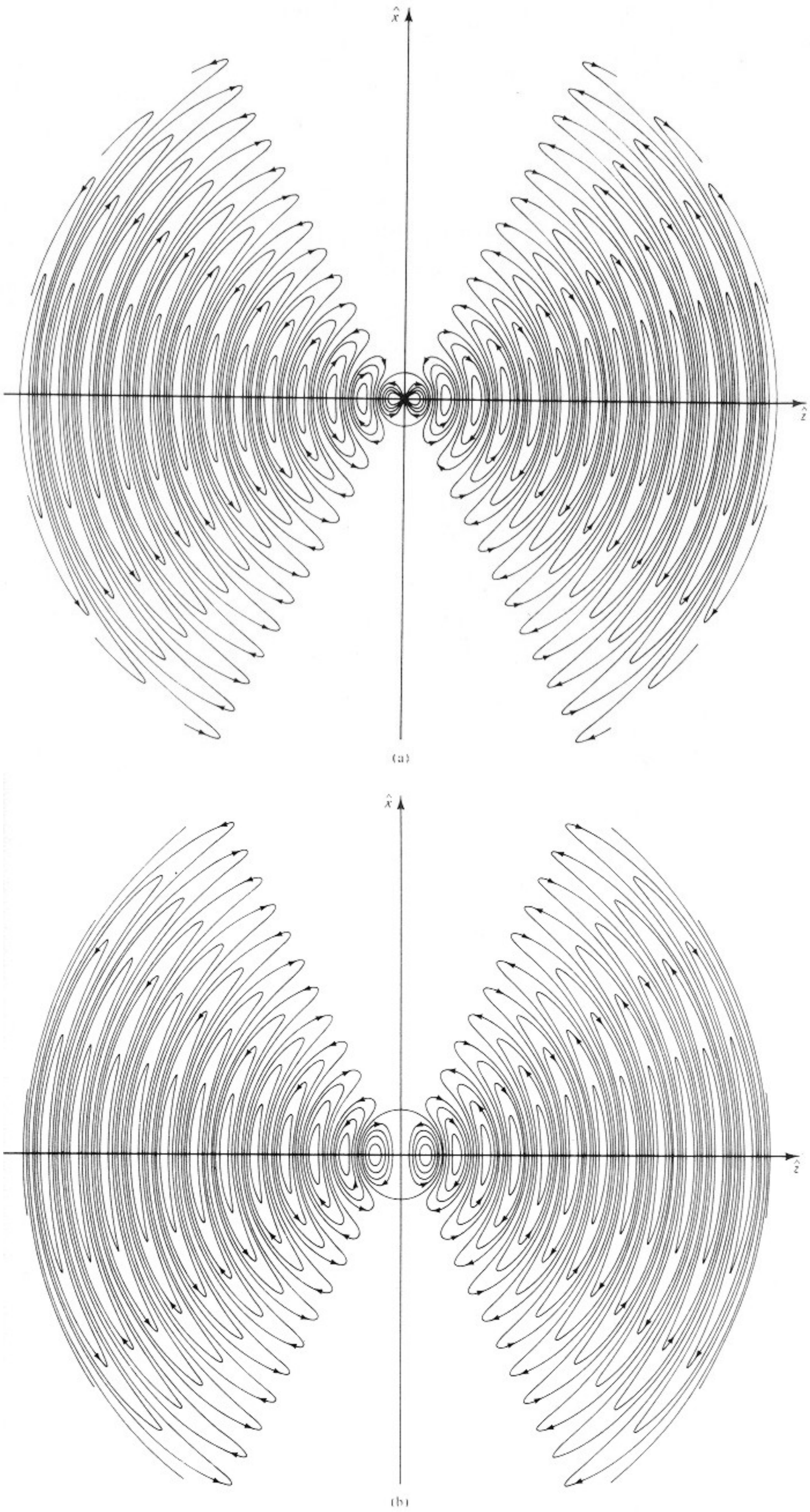


Fig. IV-4